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Sharp Estimates and Boundedness for Multilinear Integral Operators

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Abstract: In this paper, we establish a sharp inequality for some multilinear singular integral operators. As the application, we obtain the (L^p, L^q)-norm inequality for the multilinear operators.

Keywords: Multilinear operator; Singular integral operator; Sharp estimate; BMO; A_p-weight.
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1 Introduction and Results

In this paper, we consider some singular integral operators as following.

Let $T : S \to S'$ be a linear operator and there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that

$$Tf(x) = \int_{R^n} K(x,y) f(y)dy$$

for every bounded and compactly supported function f, where K satisfies: for fixed $\varepsilon > 0$ and $0 \le \delta < n$,

$$|K(x,y)| \le C|x-y|^{-n+\delta}$$

and

$$|K(y,x) - K(z,x)| + |K(x,y) - K(x,z)| \le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \le |x-z|$. T is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Let m_j be the positive integers$(j = 1, \cdots, l)$, $m_1 + \cdots + m_l = m$ and A_j be the functions on R^n $(j = 1, \cdots, l)$. The multilinear operator related to T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^m} K(x,y) f(y)dy,$$

where

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^{\alpha} A_j(y)(x-y)^{\alpha}.$$

Note that when $m = 0$, T^A is just the multilinear commutator of T and A (see [9]). While when $m > 0$, T^A is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2,3,4,5,6]). Cohen and Gosselin (see [2,3, 4])obtained the $L^p(p > 1)$ boundedness of the multilinear singular integral operator; Hu and Yang(see [8])proved a variant sharp estimate for the multilinear singular integral operators. In [12], Perez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator when $A_j \in Osc_{expL^{r_j}}$ and note that $Osc_{expL^{r_j}} \subset BMO$. The main purpose of this paper is to prove a sharp inequality for the multilinear singular integral operators when $D^{\alpha}A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$. As the application, we obtain the (L^p, L^q)-norm inequality for the multilinear operators.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [7])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c|dy.$$

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We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and $||f||_{BMO} = ||f^\#||_{L^\infty}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q\ni x} |Q|^{-1}\int_Q |f(y)|dy,$$

we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. For $1 \le p < \infty$ and $0 \le \delta < n$, let

$$M_{\delta,p}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-p\delta/n}}\int_Q |f(y)|^p dy\right)^{1/p}.$$

We denote the Muckenhoupt weights by A_p for $1 \le p < \infty$ (see [7]).

We shall prove the following theorems.

Theorem 1. Let $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\cdots,l$. Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$, $1 < r < n/\delta$ and $\tilde{x} \in R^n$,

$$(T^A(f))^\#(\tilde{x}) \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

Theorem 2. Let $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\cdots,l$. Then T^A is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$, that is

$$||T^A(f)||_{L^q} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)||f||_{L^p}.$$

2 Proof of Theorem

To prove the theorems, we need the following lemmas.

Lemma 1.(see [4]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\alpha A(z)|^q dz\right)^{1/q}$,

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2.(see [1]) Suppose that $1 \le r < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then

$$||M_{\delta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$\frac{1}{|Q|}\int_Q |T^A(f)(x) - C_0|dx \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x})$.

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0,d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j}(A_j;x,y) = R_{m_j}(\tilde{A}_j;x,y)$ and $D^\alpha\tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
&T^A(f)(x)\\
&= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)f(y)dy\\
&= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)f_2(y)dy\\
&+ \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)f_1(y)dy\\
&- \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\\
&\quad\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)f_1(y)dy\\
&- \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\\
&\quad\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)f_1(y)dy\\
&+ \sum_{|\alpha_1|=m_1,\,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\\
&\quad\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)f_1(y)dy,
\end{aligned}$$

then

$$\begin{aligned}
&\left|T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)\right| \le \left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)f_1(y)dy\right|\\
&+\left|\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)f_1(y)dy\right|\\
&+\left|\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)f_1(y)dy\right|\\
&+\left|\sum_{|\alpha_1|=m_1,\,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)f_1(y)dy\right|\\
&+\left|T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0)\right|\\
&:= I_1(x)+I_2(x)+I_3(x)+I_4(x)+I_5(x),
\end{aligned}$$

thus,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q\left|T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)\right|dx\\
\le\ &\frac{1}{|Q|}\int_Q I_1(x)dx + \frac{C}{|Q|}\int_Q I_2(x)dx + \frac{C}{|Q|}\int_Q I_3(x)dx\\
&+\frac{C}{|Q|}\int_Q I_4(x)dx + \frac{1}{|Q|}\int_Q I_5(x)dx\\
:=\ &I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 1, we get

$$R_m(\tilde{A}_j;x,y) \le C|x-y|^m\sum_{|\alpha_j|=m}||D^{\alpha_j}A_j||_{BMO},$$

thus, by the (L^r, L^q)-boundedness of T for $1 < r < n/\delta$ and $1/q = 1/r - \delta/n$, we obtain

$$\begin{aligned}
I_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\frac{1}{|Q|}\int_Q |T(f_1)(x)|dx\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_Q |T(f_1)(x)|^q dx\right)^{1/q}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1/q}\left(\int_Q |f_1(x)|^r dx\right)^{1/r}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For I_2, denoting $r = pq$ for $1 < p < n/\delta$, $q > 1$, $1/q + 1/q' = 1$ and $1/s = 1/p - \delta/n$, we have, by Hölder's inequality

$$\begin{aligned}
I_2 &\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{A}_1 f_1)(x)|dx\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1 f_1)(x)|^s dx\right)^{1/s}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1/s}\left(\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)f_1(x)|^p dx\right)^{1/p}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(x)|^{pq'}dx\right)^{1/pq'}\\
&\quad\left(\frac{1}{|Q|^{1-r\delta/n}}\int_{\tilde{Q}}|f(x)|^{pq}dx\right)^{1/pq}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3 \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

Similarly, for I_4, denoting $r = pq_3$ for $1 < p < n/\delta$, $q_1, q_2, q_3 > 1$, $1/q_1 + 1/q_2 + 1/q_3 = 1$ and $1/s = 1/p - \delta/n$, we obtain

$$\begin{aligned}
I_4 &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1)(x)|dx\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1)(x)|^s dx\right)^{1/s}\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1/s}\left(\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)f_1(x)|^p dx\right)^{1/p}\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(x)|^{pq_1}dx\right)^{1/pq_1}\\
&\quad\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(x)|^{pq_2}dx\right)^{1/pq_2}\times\left(\frac{1}{|Q|^{1-r\delta/n}}\int_{\tilde{Q}}|f(x)|^{pq_3}dx\right)^{1/pq_3}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
&T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0) =\\
&\int_{R^n}\left(\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)f_2(y)dy\\
&+\int_{R^n}\left(R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x_0,y)\right)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}K(x_0,y)f_2(y)dy\\
&+\int_{R^n}\left(R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x_0,y)\right)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}K(x_0,y)f_2(y)dy\\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\\
&\quad\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y) - \frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}K(x_0,y)\right]\\
&\quad D^{\alpha_1}\tilde{A}_1(y)f_2(y)dy\\
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\\
&\quad\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x,y) - \frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}K(x_0,y)\right]\\
&\quad D^{\alpha_2}\tilde{A}_2(y)f_2(y)dy\\
&+\sum_{|\alpha_1|=m_1,\,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}K(x_0,y)\right]\\
&\quad D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)f_2(y)dy\\
&= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.
\end{aligned}$$

By Lemma 1 and the following inequality (see [13])

$$|b_{Q_1} - b_{Q_2}| \le C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$\begin{aligned}
|R_m(\tilde{A};x,y)| &\le C|x-y|^m\sum_{|\alpha|=m}\left(||D^{\alpha}A||_{BMO} + |(D^{\alpha}A)_{\tilde{Q}(x,y)} - (D^{\alpha}A)_{\tilde{Q}}|\right)\\
&\le Ck|x-y|^m\sum_{|\alpha|=m}||D^{\alpha}A||_{BMO}.
\end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n\setminus\tilde{Q}$, we obtain, by the condition of K,

$$\begin{aligned}
|I_5^{(1)}| &\le C\int_{R^n}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{m+n+\varepsilon-\delta}}\right)\\
&\quad\prod_{j=1}^{2}|R_{m_j}(\tilde{A}_j;x,y)||f_2(y)|dy\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)\\
&\quad\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon-\delta}}\right)|f(y)|dy\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)\\
&\quad\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}\int_{2^k\tilde{Q}}|f(y)|dy\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For $I_5^{(2)}$, by the formula (see [4]):

$$R_m(\tilde{A};x,y) - R_m(\tilde{A};x_0,y) = \sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^{\beta}\tilde{A};x,x_0)(x-y)^{\beta}$$

and Lemma 1, we have

$$|R_m(\tilde{A};x,y) - R_m(\tilde{A};x_0,y)| \le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}||D^{\alpha}A||_{BMO},$$

thus

$$|I_5^{(2)}| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)$$
$$\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}|f(y)|dy$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

Similarly,

$$|I_5^{(3)}| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

For $I_5^{(4)}$, we get

$$|I_5^{(4)}|$$
$$\le C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0,y)}{|x_0-y|^m}\right|$$
$$|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||f_2(y)|dy$$
$$+C\sum_{|\alpha_1|=m_1}\int_{R^n}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|$$
$$\frac{|(x_0-y)^{\alpha_1}K(x_0,y)|}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||f_2(y)|dy$$
$$\le C\sum_{|\alpha|=m_2}||D^\alpha A_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})$$
$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{r'}dy\right)^{1/r'}$$
$$\left(\frac{1}{|2^k\tilde{Q}|^{1-r\delta/n}}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

Similarly,

$$|I_5^{(5)}| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

For $I_5^{(6)}$, taking $q_1,q_2>1$ such that $1/r+1/q_1+1/q_2=1$, then

$$|I_5^{(6)}|$$
$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m}\right|$$
$$|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f_2(y)|dy$$
$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\left(\frac{1}{|2^k\tilde{Q}|^{1-r\delta/n}}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}$$
$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{q_1}dy\right)^{1/q_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(y)|^{q_2}dy\right)^{1/q_2}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

Thus

$$|I_5| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. We choose $1<r<p$ in Theorem 1 and by using Lemma 2, we get

$$||T^A(f)||_{L^q} \le C||(T^A(f))^\#||_{L^q} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)||M_{\delta,r}(f)||_{L^q}$$
$$\le C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)||f||_{L^p}.$$

This finishes the proof.

3 Applications

In this section we shall apply Theorem 1 and 2 of the paper to several particular operators such as the Calderón-Zygmund singular integral operator and fractional integral operator.

3.1 Application 1

Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator (see [7,13,14]), the multilinear operator related to T is defined by

$$T^A(f)(x)=\int_{R^n}\frac{\prod_{j=1}^{l}R_{m_j+1}(A_j;x,y)}{|x-y|^m}K(x,y)f(y)dy,$$

Then

$$(1).\ \ (T^A(f))^\#(x) \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M_r(f)(x)$$

for any $f\in C_0^\infty(R^n)$ and $1<r<\infty$;

$$(2).\ \ ||T^A(f)||_{L^p(w)} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)||f||_{L^p(w)}$$

for any $w\in A_p$ and $1<p<\infty$.

3.2 Application 2.

Fractional integral operator with rough kernel.

For $0<\delta<n$, let T_δ be the fractional integral operator with rough kernel defined by (see [1,6])

$$T_\delta f(x)=\int_{R^n}\frac{\Omega(x-y)}{|x-y|^{n-\delta}}f(y)dy,$$

the multilinear operator related to T_δ is defined by

$$T_\delta^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^{m+n-\delta}} \Omega(x-y) f(y) dy,$$

where Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$ and $\Omega \in Lip_\varepsilon(S^{n-1})$ for some $0 < \varepsilon \le 1$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\varepsilon$. When $\Omega \equiv 1$, T is the Riesz potentials. Then

$$(3).\ (T^A(f))^\#(x) \le C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M_{\delta,r}(f)(x)$$

for any $f \in C_0^\infty(R^n)$ and $1 < r < n/\delta$;

$$(4).\ ||T^A(f)||_{L^q} \le C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) ||f||_{L^p}$$

for any $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$.

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