Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/120415

Oscillation Criteria for a Class of Nonlinear Neutral Generalized $\alpha-$Difference Equations

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Received: 2 Apr. 2018, Revised: 9 Jun. 2018, Accepted: 10 Jun. 2018
Published online: 1 Jul. 2018

Abstract: In this paper, we study the oscillation of the second-order nonlinear neutral generalized $\alpha-$difference equation of the form

$$\Delta_{\alpha(\ell)}(a(n)\Delta_{\alpha(\ell)}(x(n)+\delta p(n)x(n-\tau\ell)))+f(n,x(n-\sigma\ell))-g(n,x(n-\rho\ell))=0.$$

We obtain the oscillation criteria and provide some examples to illustrate the results.

Keywords: Generalized $\alpha-$difference operator, neutral generalized $\alpha-$difference equation, bounded oscillation, almost oscillation, bounded almost oscillation.

1 Introduction

Difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing models, statistical analysis, genetics in Biology, economics, psychology, sociology etc. These are only considered as discrete analog of differential equations several times. In fact difference equations appeared much earlier then differential equations and were instrumental in paving the way for the development of the later. Only recently, that difference equations have started receiving the attention that they deserve.

The conventional theory of difference equations is based on the operator Δ is defined as

$$\Delta[u(n)]=u(n+1)-u(n),\ n\in\mathbb{N}=\{0,1,2,3,\cdots\}. \quad (1)$$

Many authors such as Agarwal [1], Mickens [12], Elaydi [13] and Kelly [15] mentioned the same definition of the difference operator Δ and given by

$$\Delta[u(n)]=u(n+\ell)-u(n),\ n\in\mathbb{R},\ \ell\in\mathbb{R}-\{0\}, \quad (2)$$

as the incremental factor ℓ is different from unity, researchers struggled to establish new results based on the definition of Δ given in (2). Recently, Manuel et al. [11] considered operator in (2) and established some new results and new properties related to the solutions of difference equations involve Δ_ℓ, which is the notation preferred by the authors to distinguish the operators, defined in (1) and (2).

Popenda and Szmanda [4,5], introduced Δ as

$$\Delta_\alpha[u(n)]=u(n+1)-\alpha u(n) \quad (3)$$

and studied certain type of difference equations. Very recently Kilicman et al. [8,9,10] introduced the new operator as

$$\Delta_{\alpha(\ell)}[u(n)]=u(n+\ell)-\alpha u(n) \quad (4)$$

named it as the generalized α-difference operator since it generalizes the previous operators which were defined in (1), (2) and (3) and established many related results and new properties on the behavior of solutions. The generalized α-difference operator involves and generalizes many difference equations as $\alpha-$difference equation. For further study one can refer to related literature such as [3,6,7].

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Throughout this paper we use the following notations.
(a) $\mathbb{N}=\{0,1,2,3,\dots\}$, $\mathbb{N}(a)=\{a,a+1,a+2,\dots\}$,
(b) $\mathbb{N}_\ell(a)=\{a,a+\ell,a+2\ell,\dots\}$.
(c) $\lceil x\rceil$ upper integer part of x.
(d) $j=n-n_i-\left[\frac{n-n_i}{\ell}\right]\ell, n_i\in[0,\infty)$.
(e) $\lambda=\max\{\tau,\sigma,\rho\}$.

In the present study, we consider nonlinear generalized $\alpha-$difference equations having the form

$$\Delta_{\alpha(\ell)}(a(n)\Delta_{\alpha(\ell)}(x(n)+\delta p(n)x(n-\tau\ell))) + f(n,x(n-\sigma\ell))-g(n,x(n-\rho\ell))=0, \tag{5}$$

where $\delta=\pm1$, $n\ge n_0$, $a(n)>0$ with $\Delta_{\alpha(\ell)}a(n)>0$, $p(n)>0$ bounded sequence, $f(n,u)$ and $g(n,v)$ are continuous, and $\ell,\tau,\sigma,\rho\in[0,\infty)$.

Denote $\lambda=\max\{\tau,\sigma,\rho\}$, $n_1=n_0+\lambda\ell$,

$$L^1[n_0,\infty)=\left\{x(n),\sum_{s=n_0}^{\infty}|x(s)|<\infty\right\}.$$

The motivation is based on work by Wu et. al. [14] and consider the following assumptions.

(H_1) $\sum_{s=n}^{\infty}\frac{1}{\alpha^s a(s\ell)}=\infty$ for all $n\ge n_0$,
(H_2) $\frac{f(n,u)}{u}\ge q(n-\sigma\ell)>0$ for $u\ne0$ and $\frac{g(n,v)}{v}\le r(n-\rho\ell)$ for $v\ne0$,
(H_3) $0<\frac{f(n,u)}{u}\le q(n-\sigma\ell)$ for $u\ne0$ and $\frac{g(n,v)}{v}\ge r(n-\rho\ell)>0$ for $v\ne0$,
(H_4) $\frac{1}{\alpha^{\sigma-\rho}r(n)-q(n)}$ is bounded for $\sigma>\rho$,
(H_5) $\frac{1}{r(n)-\alpha^{\rho-\sigma}q(n)}$ is bounded for $\sigma<\rho$, where $r,q\in C([n_0,\infty),R^+)$.

Next, we recall the following definitions:

Definition 1.[7] *The inverse operator $\Delta^{-1}_{\alpha(\ell)}$ is defined if $\Delta_{\alpha(\ell)}v(n)=u(n)$, then*

$$\Delta^{-1}_{\alpha(\ell)}[u(n)]=v(n)-\alpha^{\left[\frac{n}{\ell}\right]}v(j) \tag{6}$$

where $n\in\mathbb{N}_\ell(j)$, $j=n-\left[\frac{n}{\ell}\right]\ell$.

In more general form is given in [11] by

$$\Delta^{-1}_{\alpha(\ell)}[u(n)]=\sum_{r=0}^{\left[\frac{n-a-j-\ell}{\ell}\right]}\frac{u(a+j+r\ell)}{\alpha^{\left\lceil\frac{a+j+\ell-n+r\ell}{\ell}\right\rceil}}+\alpha^{\left\lceil\frac{n}{\ell}\right\rceil}u(a+j), \tag{7}$$

for all $n\in\mathbb{N}_\ell(j)$, where $j=n-a-\left[\frac{n-a}{\ell}\right]\ell$.

Definition 2.[7] *The solution of (5) is called oscillatory if there exists an $n_2\in\mathbb{N}_\ell(n_1)$ for $n_1\in[a,\infty)$, such that $x(n_2)x(n_2+\ell)\le0$. If all solutions are oscillatory then equation known as oscillatory, Similarly, if not, then equation non-oscillatory (i.e. $x(n)x(n+\ell)>0$ for all $n\in[n_1,\infty)$).*

2 Oscillation Criteria of (5) when $\delta=+1$

In the next we prove some oscillation criterions when $\delta=1$ and illustrate them with suitable examples.
When $\delta=1$, the equation (5) can be rewritten as

$$\Delta_{\alpha(\ell)}(a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell))) + f(n,x(n-\sigma\ell))-g(n,x(n-\rho\ell))=0. \tag{8}$$

Theorem 1. *Let the conditions (H_1), (H_2) and (H_4) be held, and further $q(n)>r(n)$, $r(n)$ is bounded and $\sigma>\rho$. Then (8) is bounded oscillatory.*

Proof. Let $x(n)$ be a bounded and non-oscillatory. Then $x(n)$ is an eventually positive solution. That is $x(n)>0$ and $x(n-\lambda)>0$ for $n\ge n_2$. Let

$$z(n)=a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell)) -\sum_{s=\rho}^{\sigma-1}\alpha^{s-\rho}r(n-s\ell-\ell)x(n-s\ell-\ell). \tag{9}$$

From (8) and (H_2) it follows that

$$\Delta_{\alpha(\ell)}z(n)\le\left(\alpha^{\sigma-\rho}r(n-\sigma\ell)-q(n-\sigma\ell)\right)x(n-\sigma\ell)<0, \tag{10}$$

for $n\ge n_2$.
So $z(n)$ is decreasing, and $-\infty\le\lim_{n\to\infty}z(n)=c<\infty$.
If $c=-\infty$, from (9) and since $x(n)$ and $r(n)$ bounded, we have

$$\lim_{n\to\infty}a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell))=-\infty$$

Then there exists $k_1>0$ and $n_3\ge n_2$ such that

$$\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell))\le\frac{-k_1}{a(n)},\quad n\ge n_3.$$

then it follows that

$$\begin{aligned}&x(n)+p(n)x(n-\tau\ell)\\&=-k_1\alpha^{\left\lceil\frac{n-n_3}{\ell}\right\rceil}(x(n_3+j)+p(n_3+j)x(n_3+j-\tau\ell))\\&\quad-k_1\sum_{r=0}^{\left[\frac{n-n_3-\ell-j}{\ell}\right]}\frac{1}{\alpha^{\left\lceil\frac{n_3+j+\ell-n+r\ell}{\ell}\right\rceil}a(n_3+j+\ell+r\ell)}\end{aligned}$$

As $n\to\infty$, according to (H_1), we obtain

$$\lim_{n\to\infty}(x(n)+p(n)x(n-\tau\ell))=-\infty.$$

That is a contradiction with considering $x(n)$ and $q(n)$ bounded. Thus we have $|c| < \infty$, and $z(n)$ is bounded. From (10) it follows that

$$x(n-\sigma\ell) \le \frac{1}{\alpha^{\sigma-\rho}r(n-\sigma\ell)-q(n-\sigma\ell)}\Delta_{\alpha(\ell)}z(n). \quad (11)$$

So, $x \in L^1[n_0,\infty)$ by (H_4).
(i) If $c > 0$, using (9) we obtain

$$z(n) \le a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell)), \quad n \ge n_2.$$

Now, since $z(n) \to c$ as $n \to \infty$, leads to

$$\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell)) \ge \frac{c}{a(n)}, \quad n \ge n_2$$

and

$$\begin{aligned} &x(n)+p(n)x(n-\tau\ell) \\ &= c\alpha^{\left\lceil\frac{n-n_2}{\ell}\right\rceil}(x(n_2+j)+p(n_2+j)x(n_2+j-\tau\ell)) \\ &\quad + c\sum_{r=0}^{\left[\frac{n-n_2-\ell-j}{\ell}\right]}\frac{1}{\alpha^{\left\lceil\frac{n_2+j+\ell-n+r\ell}{\ell}\right\rceil}a(n_2+j+\ell+r\ell)}. \end{aligned}$$

On using (H_1) then $\lim_{n\to\infty}(x(n)+p(n)x(n-\tau\ell)) = \infty$, that contradicts with assumption that $x(n)$ is bounded.

(ii) If $c < 0$, and $x \in L^1[n_0,\infty)$, we have

$$\lim_{n\to\infty}\sum_{s=\rho}^{\sigma-1}\alpha^{s-\rho}r(n-s\ell-\ell)x(n-s\ell-\ell) = 0.$$

Then, since $z(n) \to c$ as $n \to \infty$, there exists $\varepsilon \in (0,-c)$ and $n_4 \ge n_2$ such that

$$a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell)) \le c+\varepsilon < 0, \quad n \ge n_4$$

and

$$\begin{aligned} &x(n)+p(n)x(n-\tau\ell) \\ &= (c+\varepsilon)\alpha^{\left\lceil\frac{n-n_4}{\ell}\right\rceil}(x(n_4+j)+p(n_4+j)x(n_4+j-\tau\ell)) \\ &\quad +(c+\varepsilon)\sum_{r=0}^{\left[\frac{n-n_4-\ell-j}{\ell}\right]}\frac{1}{\alpha^{\left\lceil\frac{n_4+j+\ell-n+r\ell}{\ell}\right\rceil}a(n_4+j+\ell+r\ell)} \end{aligned}$$

Hence, by (H_1) again, we obtain

$$\lim_{n\to\infty}(x(n)+p(n)x(n-\tau\ell)) = -\infty,$$

that is also a contradiction by assuming that $x(n)$ and $q(n)$ bounded.

(iii) If $c = 0$, and $\Delta_{\alpha(\ell)}z(n) < 0$, then we have $z(n) > 0$. So

$$\begin{aligned} &a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell)) \\ &> \sum_{s=\rho}^{\sigma-1}\alpha^{s-\rho}r(n-(s+1)\ell)x(n-(s+1)\ell) > 0, \quad n \ge n_2. \end{aligned}$$

Since $x(n)+p(n)x(n-\tau\ell)$ is positive and increasing, the summation

$$\sum_{s=n_0}^{\infty}(x(s)+p(s)x(s-\tau\ell))$$

does not converge, that is a contradiction again saying that $x \in L^1[n_0,\infty)$. These contradictions show that (8) has no bounded positive solution. Now, suppose conversely that $x(n)$ has a bounded eventually negative solution. Then $x(n-\lambda\ell) < 0$ for some $n_2 > n_1$ and all $n \ge n_2$. From (8), (9) and (H_1), we have

$$\Delta_{\alpha(\ell)}z(n) \ge (\alpha^{\sigma-\rho}r(n-\sigma\ell)-q(n-\sigma\ell))x(n-\sigma\ell) > 0, \quad n \ge n_2. \quad (12)$$

Thus, $z(n)$ is increasing and $-\infty < \lim_{n\to\infty}z(n) = c \le \infty$. That is every bounded solution of (8) is oscillatory.

*Example 1.*Consider

$$\begin{aligned} &\Delta_{\alpha(\ell)}\left(n\Delta_{\alpha(\ell)}(x(n)+2x(n-2\ell))\right) \\ &\quad +3n(1+\alpha)^2x(n-3\ell)-3(\ell+\alpha\ell)x(n-2\ell) = 0 \quad (13) \end{aligned}$$

viewing (13) as (8), we have $a(n) = n$, $p(n) = 2 > 0$, and

$$q(n) = 3(1+\alpha)^2(n+3\ell) > r(n) = 3\ell(1+\alpha) > 0.$$

Further, $\tau = 2$, $\sigma = 3$, $\rho = 2$ and $r(n)$ is bounded for $n \ge 3\ell$. That is the theorem 1 is held and the equation is bounded oscillatory. In fact, $x(n) = (-1)^{\left\lceil\frac{n}{\ell}\right\rceil}$ is one such solution.

Theorem 2.*Assume (H_1), (H_2) and (H_5) is held $q(n) > r(n)$, and $q(n)$, $\frac{1}{a(n)}$ are bounded and $\sigma < \rho$. Then (8) is almost oscillatory.*

*Proof.*Suppose that $x(n)$ is an eventually positive solution. Take $n_2 \ge n_1$ such that $x(n-\lambda\ell) > 0$ for all $n \ge n_2$. Let

$$\begin{aligned} z(n) &= a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell)) \\ &\quad + \sum_{s=\sigma}^{\rho-1}\alpha^{s-\sigma}q(n-(s+1)\ell)x(n-(s+1)\ell). \quad (14) \end{aligned}$$

From (8) it follows that

$$\Delta_{\alpha(\ell)}z(n) \le \left(r(n-\rho\ell)-\alpha^{\rho-\sigma}q(n-\rho\ell)\right)x(n-\rho\ell) < 0, \quad n \ge n_2. \quad (15)$$

So, $z(n)$ is decreasing and

$$-\infty \le \lim_{n\to\infty}z(n) = c < \infty.$$

If $c = -\infty$, then

$$\lim_{n\to\infty}a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell)) = -\infty$$

On using (H_1), we obtain

$$\lim_{n\to\infty}(x(n)+p(n)x(n-\tau\ell)) = -\infty,$$

which contradicts $(x(n)+p(n)x(n-\tau\ell))>0$. Therefore $|c|<\infty$ so $z(n)$ is bounded.
From (15) we have

$$x(n-\rho\ell) \leq \frac{1}{r(n-\rho\ell)-\alpha^{\rho-\sigma}q(n-\rho\ell)}\Delta_{\alpha(\ell)}z(n), \quad n \geq n_2, \tag{16}$$

so, by (H_5), $x \in L^1[n_0,\infty)$ and

$$\lim_{n\to\infty}\sum_{s=\sigma}^{\rho-1}\alpha^{s-\sigma}q(n-(s+1)\ell)x(n-(s+1)\ell)=0.$$

Since $\dfrac{1}{a(s)}$ is bounded, by (14), thus

$$\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell))$$

is bounded. This implies that $(x(n)+p(n)x(n-\tau\ell))$ is convergent on $[n_1,\infty)$. Note that the property $x \in L^1[n_0,\infty)$ and boundedness of $q(n)$ implies that

$$(x(n)+p(n)x(n-\tau\ell)) \in L^1[n_0,\infty).$$

Hence

$$\lim_{n\to\infty}(x(n)+p(n)x(n-\tau\ell))=0 \Rightarrow \lim_{n\to\infty}x(n)=0.$$

Therefore, every solution $x(n)$ of (8) which is not $o(1)$ as $n\to\infty$, it is oscillatory.

*Example 2.*Let

$$\Delta_{\alpha(\ell)}\left(\Delta_{\alpha(\ell)}\left(x(n)+\frac{n}{n-4\ell}x(n-4\ell)\right)\right) + \frac{2n}{n-\ell}(1+2\alpha+\alpha^2)x(n-\ell)-\frac{4\ell(1+\alpha)}{n-2\ell}x(n-2\ell)=0. \tag{17}$$

By (17) as (8), we have $a(n)=1$, $p(n)=\dfrac{n}{n-4\ell}>0$, and

$$q(n)=\frac{2(n+\ell)(1+\alpha)^2}{n} > r(n)=\frac{4\ell(1+\alpha)}{n}>0.$$

Moreover, $\tau=4$, $\sigma=1$, $\rho=2$ and $q(n)$ is bounded for $n\geq 5\ell$. Then the conditions of theorem 1 are satisfied so the equation is almost oscillatory. Clearly, $x(n)=n(-1)^{\lceil \frac{n}{\ell}\rceil}$ is one such solution.

Theorem 3.*Suppose (H_1), (H_3) and (H_4) hold, $q(n)<r(n)$, $r(n)$, $\dfrac{1}{a(n)}$ are bounded and $\sigma>\rho$. Then (8) is bounded almost oscillatory.*

*Proof.*Assume that $x(n)$ is a bounded positive solution and $z(n)$ is defined by (9). Take $n_2\geq n_1$ such that $x(n-\lambda\ell)>0$ for all $n\geq n_2$. From (8) and (H_3), we have

$$\Delta_{\alpha(\ell)}z(n) \geq \big(\alpha^{\sigma-\rho}r(n-\sigma\ell)-q(n-\sigma\ell)\big)x(n-\sigma\ell)<0, \quad n\geq n_2. \tag{18}$$

So, $z(n)$ increases. Then

$$-\infty \leq \lim_{n\to\infty}z(n)=d\leq\infty.$$

If $\lim_{n\to\infty}z(n)=\infty$, then from (9) and the boundedness of $x(n)$ and $r(n)$, we obtain

$$\lim_{n\to\infty}a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell))=\infty.$$

Then there exists $l_2>0$ and $n_3\geq n_2$ such that

$$a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell))\geq l_2, \quad n\geq n_3.$$

From (H_1) it follows that

$$\lim_{n\to\infty}(x(n)+p(n)x(n-\tau\ell))=\infty,$$

this a contradiction by considering $x(n)$ and $q(n)$ bounded, so $|d|<\infty$ and $z(n)$ is bounded. From (18) we have

$$x(n-\sigma\ell) \leq \frac{1}{\alpha^{\sigma-\rho}r(n-\sigma\ell)-q(n-\sigma\ell)}\Delta_{\alpha(\ell)}z(n), \quad n\geq n_2.$$

Therefore, by (H_4), $x\in L^1[n_0,\infty)$. Similar to the proof of theorem 2, we have $\lim_{n\to\infty}x(n)=0$. That is $x(n)$ in (8) not $o(1)$ class as $n\to\infty$ thus it is oscillatory.

Theorem 4.*Suppose (H_1), (H_3) and (H_5) hold, $q(n)<r(n)$, $q(n)$, $\dfrac{1}{a(n)}$ are bounded and $\sigma<\rho$. Then (8) is bounded almost oscillatory.*

*Proof.*Similar to the previous proofs let $z(n)$ be defined by (14). Take $n_2\geq n_1$ such that $x(n-\lambda\ell)>0$ for $n\geq n_2$. From (8) and (H_3), we have

$$\Delta_{\alpha(\ell)}z(n) \geq \big(r(n-\rho\ell)-\alpha^{\rho-\sigma}q(n-\rho\ell)\big)x(n-\rho\ell)>0, \quad n\geq n_2. \tag{19}$$

Hence $z(n)$ increases and

$$-\infty<\lim_{n\to\infty}z(n)=d\leq\infty.$$

By using proof of theorem 3, we have $-\infty<d<\infty$. Therefore, $z(n)$ is bounded. From (19) it follows that

$$x(n-\rho\ell) \leq \frac{1}{r(n-\rho\ell)-\alpha^{\rho-\sigma}q(n-\rho\ell)}\Delta_{\alpha(\ell)}z(n), \quad n\geq n_2.$$

Thus, by (H_5), $x\in L^1[n_0,\infty)$ and

$$\lim_{n\to\infty}\sum_{s=\sigma}^{\rho-1}\alpha^{s-\sigma}q(n-(s+1)\ell)x(n-(s+1)\ell)=0.$$

Then it follows from (14) that

$$\lim_{n\to\infty}a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell))=d.$$

(i) If $d > 0$, then there exists $n_5 \geq n_2$ such that

$$a(n)\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell)) \geq \frac{d}{2}, \quad n \geq n_5. \quad (20)$$

From condition (H_1) we have

$$\lim_{n\to\infty}(x(n)+p(n)x(n-\tau\ell)) = \infty,$$

which is a contradiction.

(ii) If $d < 0$, as in the case (i), we have

$$\lim_{n\to\infty}(x(n)+p(n)x(n-\tau\ell)) = -\infty,$$

it follows that a contradiction $x(n)$ and $q(n)$ assuming bounded.
Hence $d = 0$, i.e., $\lim_{n\to\infty} z(n) = 0$. Now, in view of (14),

$$\Delta_{\alpha(\ell)}(x(n)+p(n)x(n-\tau\ell)) < 0 \Rightarrow x(n)+p(n)x(n-\tau\ell).$$

Since

$$x(n)+p(n)x(n-\tau\ell) \in L^1[n_0,\infty),$$

then it follows that

$$\lim_{n\to\infty}(x(n)+p(n)x(n-\tau\ell)) = 0 \Rightarrow \lim_{n\to\infty} x(n) = 0.$$

Therefore, $x(n)$ is not in the class of $o(1)$ as $n\to\infty$ thus it must be oscillatory.

*Example 3.*Let

$$\Delta_{\alpha(\ell)}\left(n\Delta_{\alpha(\ell)}(x(n)+x(n-2\ell))\right)+2\alpha^3\left(1+\alpha^{-2}\right)x(n-\ell) \\ -4n\alpha^6\left(1+\alpha^{-2}\right)x(n-4\ell) = 0 \quad (21)$$

then it follows that we have $a(n) = n$, $p(n) = 1 > 0$, and

$$q(n) = 2\alpha^3\left(1+\alpha^{-2}\right) < r(n) = 4n\alpha^6\left(1+\alpha^{-2}\right).$$

Moreover, $\tau = 2$, $\sigma = 1$, $\rho = 4$ and $q(n)$ is bounded for $n \geq 5\ell$. Then by theorem 1 we obtain

$$x(n) = (-\alpha)^{\lceil \frac{n}{\ell} \rceil}$$

which is such a solution.

3 Oscillation Criteria of (5) when $\delta = -1$

In this section, we consider (5) when $\delta = -1$, so (5) becomes

$$\Delta_{\alpha(\ell)}(a(n)\Delta_{\alpha(\ell)}(x(n)-p(n)x(n-\tau\ell))) \\ +f(n,x(n-\sigma\ell))-g(n,x(n-\rho\ell)) = 0. \quad (22)$$

Theorem 5.*Suppose (H_1), (H_3) and (H_4) hold, $p(n) \geq 1$, $q(n) < r(n)$, $\sigma < \rho$ and $r(n)$ is bounded. Then (22) is bounded oscillatory.*

*Proof.*Suppose $x(n)$ is a bounded non-oscillatory that implies $x(n)$ is an eventually positive solution. Now let

$$z(n) = a(n)\Delta_{\alpha(\ell)}(x(n)-p(n)x(n-\tau\ell)) \\ +\sum_{s=\rho}^{\sigma-1}\alpha^{s-\rho}r(n-(s+1)\ell)x(n-(s+1)\ell). \quad (23)$$

From the theorem 3, we obtain

$$\Delta_{\alpha(\ell)}z(n) > 0, \ \lim_{n\to\infty} z(n) = c, \ |c| < \infty, \ \text{and } x \in L^1[n_0,\infty).$$

(i) If $c > 0$, from (23) it follows that

$$\lim_{n\to\infty} a(n)\Delta_{\alpha(\ell)}(x(n)-p(n)x(n-\tau\ell)) = c > \frac{c}{2}.$$

So, for large enough n

$$\Delta_{\alpha(\ell)}(x(n)-p(n)x(n-\tau\ell)) \geq \frac{c}{2a(n)}.$$

By condition (H_1), we have

$$\lim_{n\to\infty}(x(n)-p(n)x(n-\tau\ell)) = \infty,$$

which is a contradictions.

(ii) If $c < 0$, in view of $\lim_{n\to\infty} z(n) = c$ and $x \in L^1[n_0,\infty)$, there exists $n_6 \geq n_1$ such that

$$a(n)\Delta_{\alpha(\ell)}(x(n)-p(n)x(n-\tau\ell)) \leq \frac{c}{2} < 0, \quad n \geq n_6.$$

By (H_1), we have $\lim_{n\to\infty}(x(n)-p(n)x(n-\tau\ell)) = -\infty$, which is also a contradiction.

(iii) If $c = 0$, in view of $\Delta_{\alpha(\ell)}z(n) > 0$, we have $z(n) < 0$. Further,

$$\Delta_{\alpha(\ell)}(x(n)-p(n)x(n-\tau\ell)) < 0.$$

We show that $x(n)-p(n)x(n-\tau\ell) > 0$. In fact, if there exists $n_7 \geq n_1$ such that $(x(n_7)-p(n_7)x(n_7-\tau\ell)) < 0$, then, for all $n \geq n_7$,

$$x(n)-p(n)x(n-\tau\ell) \leq x(n_7)-p(n_7)x(n_7-\tau\ell) < 0.$$

This contradicts $x(n)-p(n)x(n-\tau\ell) \in L^1[n_0,\infty)$.
Hence, $x(n)-p(n)x(n-\tau\ell) > 0$ for all large $n \geq n_1$. From this and the assumption on p, we have

$$x(n) \geq p(n)x(n-\tau\ell) \geq x(n-\tau\ell),$$

which contradicts $x \in L^1[n_0,\infty)$. Thus (22) is bounded oscillatory.

*Example 4.*Given generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\left(\frac{1}{n}\Delta_{\alpha(\ell)}\left(x(n)-2x(n-2\ell)\right)\right)+\frac{(1+\alpha)}{n+\ell}x(n-\ell) \\ -\frac{\alpha^2+\alpha}{n}x(n-4\ell) = 0. \quad (24)$$

Viewing (24) as (22), we have $a(n) = \frac{1}{n}$, $p(n) = 2 > 0$, and

$$q(n) = \frac{(1+\alpha)}{n+\ell} < r(n) = \frac{(1+\alpha)\alpha}{n}.$$

By letting, $\tau = 2$, $\sigma = 1$, $\rho = 4$ and $r(n)$ bounded for $n \geq 5\ell$. Then the generalized $\alpha-$difference equation is bounded almost oscillatory. Clearly $x(n) = (-1)^{\lceil \frac{n}{\ell} \rceil}$ is one such solution.

Theorem 6. *Suppose (H_1), (H_3) and (H_5) hold, $q(n) < r(n)$, $\sigma > \rho$, $0 \leq p(n) \leq p_1 < 1$ or $1 < p_2 \leq p(n)$, $r(n)$ and $\frac{1}{a(n)}$ are bounded. Then (22) is bounded almost oscillatory.*

Proof. The proof is similar to the proof of theorem 3, thus we obtain

$$\lim_{n\to\infty}(x(n) - p(n)x(n-\tau\ell)) = 0.$$

Suppose

$$\limsup_{n\to\infty} x(n) = k > 0.$$

So, there exists $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} x(n_k) = k > 0.$$

(i) If $0 \leq p(n) \leq p_1 < 1$, then we have $(1-p_1)k \leq 0$, which contradicts $k > 0$ and $1 - p_1 > 0$.
(ii) If $1 < p_2 \leq p(n)$, then we have $0 \leq (1-p_2)k$, which contradicts $k > 0$ and $p_2 - 1 > 0$. Therefore, we must have

$$\limsup_{n\to\infty} x(n) = 0$$

Then $\lim_{n\to\infty} x(n) = 0$ as $x(n)$ is eventually positive. This shows that (22) is bounded almost oscillatory.

Theorem 7. *Suppose (H_1), (H_2) and (H_4) hold, $q(n) > r(n)$, $\sigma < \rho$, $0 \leq p(n) \leq p_1 < 1$ or $1 < p_2 \leq p(n)$, $q(n)$ and $\frac{1}{a(n)}$ are bounded. Then (22) is bounded almost oscillatory.*

Proof. The proof is similar to theorem 2, we obtain

$$\limsup_{n\to\infty}(x(n) - p(n)x(n-\tau\ell)) = 0.$$

Then followed by proof of theorem 6.

Theorem 8. *Suppose that conditions (H_1), (H_2) and (H_4) hold, $q(n) > r(n)$, $\sigma > \rho$, $q(n)$ is bounded, $0 \leq p(n) \leq p_1 < 1$ or $\frac{1}{a(n)}$ is bounded and $1 < p_2 \leq p(n)$. Then (22) is bounded almost oscillatory.*

Proof. Let

$$z(n) = a(n)\Delta_{\alpha(\ell)}(x(n) - p(n)x(n-\tau\ell)) - \sum_{s=\rho}^{\sigma-1} \alpha^{s-\rho} r(n-s\ell-\ell)x(n-s\ell-\ell). \quad (25)$$

Then we have $x \in L^1[n_0,\infty)$, $\lim_{n\to\infty} z(n) = c = 0$ and $\Delta_{\alpha(\ell)} z(n) < 0$. So, $z(n) > 0$,

$$\Delta_{\alpha(\ell)}(x(n) - p(n)x(n-\tau\ell)) > 0$$

and $(x(n) - p(n)x(n-\tau\ell))$ increases. Thus

$$x(n) - p(n)x(n-\tau\ell) < 0$$

for $n \geq n_1$. In fact, if there exists $n_8 \leq n_1$ such that $x(n_8) - p(n_8)x(n_8 - \tau\ell) \geq 0$, then

$$x(n) - p(n)x(n-\tau\ell) \geq x(n_8+\ell) - p(n_8+1)x(n_8+\ell-\tau\ell) > 0$$

for $n \geq n_8 + \ell$, which contradicts $x(n) - p(n)x(n-\tau\ell) \in L^1[n_0,\infty)$. Hence $x(n) - p(n)x(n-\tau\ell) < 0$ for all $n \geq n_1$. If $0 \leq p(n) \leq p_1 < 1$ is satisfied, then $x(n) < p(n)x(n-\tau\ell)$ for all $n \geq n_1$. This implies that $\lim_{n\to\infty} x(n) = 0$.
If $\frac{1}{a(n)}$ is bounded and $1 < p_2 \leq p(n)$, from the proof of theorem 6, we have

$$\lim_{n\to\infty}(x(n) - p(n)x(n-\tau\ell)) = 0$$

and thus $\lim_{n\to\infty} x(n) = 0$. Therefore, (22) is bounded almost oscillatory.

Example 5. Consider the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\left(\frac{n}{n+\ell}\Delta_{\alpha(\ell)}(x(n) - 2x(n-2\ell))\right) + (2\alpha^2 - 4)\frac{n+\ell}{n+2\ell}x(n-3\ell) - \frac{2n(\alpha^2-2}{n+\ell}x(n-2\ell) = 0. \quad (26)$$

Viewing (26) as (22), we have $a(n) = \frac{n}{n+\ell}$, $p(n) = 2 > 0$, and

$$q(n) = \frac{2\alpha^3(\alpha^2-2)(n+4\ell)}{n+5\ell} > r(n) = \frac{(2\alpha^4 - 4\alpha^2)(n+2\ell)}{n+3\ell}.$$

Moreover, $\tau = 2$, $\sigma = 3$, $\rho = 2$ and $q(n)$ is bounded for $n \geq 3\ell$. All the conditions of theorem 8 hold and hence generalized $\alpha-$difference equation is bounded almost oscillatory. In fact $x(n) = (-\alpha)^{\lceil \frac{n}{\ell} \rceil}$ is one such solution.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, Y. H Kim and A. K. Sen, New Retarded Discrete Inequalities with Applications, International Journal of Difference Equations, 1(4), 1–19 (2009).

[3] A. Akgul and A. Kılıçman, Solving Delay Differential Equations by an Accurate Method with Interpolation, Abstract and Applied Analysis, Volume 2015, Article ID 676939, 7 pages.

[4] J. Popenda and B. Szmanda, On the Oscillation of Solutions of Certain Difference Equations, Demonstratio Mathematica, XVII(1), (1984), 153–164.

[5] J. Popenda, Oscillation and Nonoscillation Theorems for Second-Order Difference Equations, Journal of Mathematical Analysis and Applications, 123(1), (1987), 34–38.

[6] M. M. S. Manuel and G. B. A. Xavier, Generalized Difference Calculus of Sequences of Real and Complex Numbers, International Journal of Computational and Numerical Analysis and applications 6(4), 401–415 (2004).

[7] M. M. S. Manuel, G. B. A. Xavier, D. S. Dilip and G. D. Babu, Oscillation, Nonoscillation and Growth of Solutions of Generalized Second Order Nonlinear α-Difference Equations, Global Journal of Mathematical Sciences: Theory and Practical, 4(1), 211–225(2012).

[8] M. M. S. Manuel, A. Kılıçman, G.B.A. Xavier, P. Pugalarasu and D. S. Dilip, On The Solution of Second Order Generalized Difference Equations, Advances in Difference Equations, 2012, 105 (2012).

[9] M. M. S. Manuel, A. Kılıçman, G.B.A. Xavier, P. Pugalarasu and D. S. Dilip, An Application on The Second Order Generalized Difference Equations, Advances in Difference Equations, 2013, 35 (2013).

[10] M. M. S. Manuel, A. Kılıçman, K. Srinivasan and D. G. Babu, Oscillation of Solution of Some Generalized Nonlinear $\alpha-$Difference Equations, Advances in Difference Equations, 2014, 109 (2014).

[11] M. M. S. Manuel, G.B.A. Xavier and E. Thandapani, Theory of Generalized Difference Operator and Its Applications, Far East Journal of Mathematical Sciences, 20(2), pp. 163–171 (2006).

[12] R. E. Mickens, Difference Equations, Van Nostrand Reinhold Company, New York, 1990.

[13] S. N. Elaydi, An Introduction To Difference Equations, Second Edition, Springer, 1999.

[14] S. Wu, P. S. Calay and Z. Hou, Oscillation Criteria for a Class of Nonlinear Neutural Differential Equations, Advances in Difference Equations, 2015: 154 (2015)

[15] W. G. Kelley, A. C. Peterson, Difference Equations: An Introduction with Applications, Academic Press, 1991.

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