Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/120403

A Mixed Problem with Integral Condition for a Parabolic System

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Received: 4 Jan. 2018, Revised: 27 Feb. 2018, Accepted: 7 Mar. 2018
Published online: 1 Jul. 2018

Abstract: In this paper, we use a functional analysis approach to prove the well-posedness for a parabolic system with a boundary Neumann condition and a nonlocal integral condition. The last step, is to establish the density of the range $R(L)$ in the Hilbert space H, and hence the existence of a strong solution of the problem. The proof is mainly based on some a priori bounds and on some density arguments.

Keywords: Parabolic equation, boundary integral conditions, energy inequality

1 Introduction

Many physical phenomena can be modeled and described in terms of local problems, such as mixed problems with boundary classical conditions. More precisely standard (Dirichlet, Neumann type) conditions which are prescribed pointwise are not always adequate as it depends on the physical context which data can be measured at the boundary of the physical domain. In some cases it is not possible to prescribe the solution u (pressure, temperature,...) point-wise, because only the average value of the solution can be measured along the boundary or along some part of it. Certain of these problems are described by parabolic initial-boundary value problems in one space variable with nonstandard boundary conditions that involve an integral term over the spatial domain of a function of the desired solution. This category of equations represents a large class of commonly occurring problems in mathematical physics and engineering, see e.g. [9,10].

Using diferent methods, mixed problems with classical and nonlocal (integral) boundary conditions related to parabolic and hyperbolic equations have been extensively investigated and several results concerning existence and uniqueness have been established. For the parabolic case, using the potential method: we cite here Cannon [4], Kamynin [14]. Fourier method: Ionkin [12], Ionkin and Moiseev [13]. And the energy method: we cite Bouziani [3], Mesloub [18] Kartynik [15], Mesloub and Bouziani [19,21], Mesloub and Lekrine [22], and Yurchuk [30]. For the hyperbolic case, we cite Mesloub and Bouziani [20], Muravei and Philinovskii [27], Pulkina [28,29].

In this work, we prove the well-posedness for a parabolic system with a classical Neumann boundary condition and a nonlocal integral condition. The last step, is to establish the density of the range $R(L)$ in the space H, and hence the existence of a strong solution of the problem.

2 Position of the problem

In the bounded domain

$$Q = \Omega \times (0,T) = \left\{(x,t) : 0 < x < l,\ 0 < T < \infty, l \in IR_{+}^{*}\right\},$$

we consider the following system

$$\mathscr{L}_1 u = u_t - (a(x,t)u_x)_x - (b(x,t)v_x)_x = f(x,t), \tag{1}$$

$$\mathscr{L}_2 v = v_t - (c(x,t)v_x)_x - k(x,t)u = g(x,t), \tag{2}$$

where the functions $a(x,t), b(x,t), c(x,t), k(x,t)$, satisfy the conditions

$$P_1 : a(0,t) = b(0,t) = c(0,t) = 0,$$

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$$P_2 : \begin{Bmatrix} a_0 \le a(x,t) \le a_1, 0 \le a_x(x,t) \le a_2, \\ a_3 \le a_t(x,t) \le a_4, 0 \le a_{tt}(x,t) \le a_5, \\ \\ 0 \le b(x,t) \le b_1, 0 \le b_x(x,t) \le b_2, \\ 0 \le b_t(x,t) \le b_3, \\ \\ c_0 \le c(x,t) \le c_1, 0 \le c_x(x,t) \le c_2, \\ c_3 \le c_t(x,t) \le c_4, 0 \le c_{tt}(x,t) \le c_5, \\ \\ 0 \le k(x,t) \le k_1, 0 \le k_t(x,t) \le k_2, \end{Bmatrix}$$

for all (x,t) in Q, where

$a_i \in IR_*^+, \forall i = \overline{0,5}, b_k \in IR_*^+, \forall k = \overline{1,3}$,
$c_j \in IR_*^+, \forall j = \overline{0,5}, k_h \in IR_*^+, \forall h = \overline{1,2}$.

To equations (1) and (2) we associate the initial conditions

$$\ell_1 u = u(x,0) = u_0(x), \qquad 0 < x < l, \tag{3}$$

$$\ell_2 v = v(x,0) = v_0(x), \qquad 0 < l < 1, \tag{4}$$

the Neumann boundary conditions

$$u_x(l,t) = 0, \qquad 0 < t < T, \tag{5}$$

$$v_x(l,t) = 0, \qquad 0 < t < T, \tag{6}$$

and the integral conditions

$$\int \int_0^l u(x)dx = \int \int_0^l v(x)dx = 0. \tag{7}$$

The functions u_0 and v_0 satisfy the conditions

$$\frac{\partial u_0}{\partial x}(l) = \frac{\partial v_0}{\partial x}(l) = 0, \int \int_0^l u_0(x)dx = \int \int_0^l v_0(x)dx = 0. \tag{8}$$

3 Functional frame

For the investigation of the posed problem (1)-(8), we introduce some function spaces that we extensively use in the sequel.

Let $L^2(Q)$, where $Q = \Omega \times (0,T)$ be the weighted Hilbert space of square integrable functions having the finite norm

$$\|u\|^2_{L^2(Q)} = \int_Q u^2 dxdt.$$

The inner product in $L^2(Q)$ is defned by

$$(u,v)_{L^2(Q)} = \int_Q uvdxdt.$$

Problem (1)-(8) can be put in the operational form:

$$AU = \mathscr{H}, \tag{9}$$

where in this last equation

$$U = (u,v), \tag{10}$$

$$AU = (L_1 u, L_2 v), \tag{11}$$

and

$$\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2), \tag{12}$$

where (11) and (12) are defined respectively by

$$L_1 u = \{\mathscr{L}_1 u, \ell_1 u\}, \quad L_2 v = \{\mathscr{L}_2 v, \ell_2 v\}, \tag{13}$$

and

$$\mathscr{H}_1 = \{f, u_0\}, \quad \mathscr{H}_2 = \{g, v_0\}. \tag{14}$$

The operator A is considered from $B = B_1 \times B_2$ into $H = H_1 \times H_2$, where B is the Banach space functions $(u,v) \in (L^2(Q))^2$ satisfying conditions (6)-(8) and having the finite norm

$$\|U\|^2_B = \|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|\Im_x u\|^2_{L^2(\Omega)} + \|\Im_x v\|^2_{L^2(\Omega)}, \tag{15}$$

and $H = H_1 \times H_2$ is the Hilbert space $(L^2(Q))^2 \times (L^2(Q))^2$, with the finite norm

$$\|\mathscr{H}\|^2_H = \|f\|^2_{L^2(Q)} + \|g\|^2_{L^2(Q)} + \|u_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega)}. \tag{16}$$

We define the domain of definition $D(A)$ of the operator A to be the set

$$D(A) = \left\{(u,v) \in (L^2(Q))^2 / \ u_t, \ v_t, \ u_x, \ v_x, u_{tx}, v_{tx} \in L^2(Q)\right\},$$

satisfying conditions (3)-(8).

4 Uniqueness of solution

We may state the following result.

Theorem 1. *Let $a(x,t), b(x,t)$ and $c(x,t)$ satisfying conditions P_1 and P_2. For all function $U = (u,v) \in D(A)$ there exists a positive constant C independent of U, such that*

$$\|U\|_B \le C \|AU\|_H. \tag{17}$$

Proof. We consider the scalar products

$$(u - \Im_x^2 u, \mathscr{L}_1 u)_{L^2(Q^\tau)} \text{ and } (v - \Im_x^2 v, \mathscr{L}_2 \theta)_{L^2(Q^\tau)}, \tag{18}$$

where $Q^\tau = \Omega \times (0,\tau)$, then we have

$$\begin{aligned} &(u,u_t)_{L^2(Q^\tau)} - (u,(a(x,t)u_x)_x)_{L^2(Q^\tau)} \\ &-(u,(b(x,t)v_x)_x)_{L^2(Q^\tau)} - (\Im_x^2 u, u_t)_{L^2(Q^\tau)} \\ &+(\Im_x^2 u,(a(x,t)u_x)_x)_{L^2(Q^\tau)} + (\Im_x^2 u,(b(x,t)v_x)_x)_{L^2(Q^\tau)} \\ &+(v,v_t)_{L^2(Q^\tau)} - (v,(c(x,t)v_x)_x)_{L^2(Q^\tau)} \\ &-(v,k(x,t)u)_{L^2(Q^\tau)} - (\Im_x^2 v, v_t)_{L^2(Q^\tau)} \\ &+(\Im_x^2 v,(c(x,t)v_x)_x)_{L^2(Q^\tau)} + (\Im_x^2 v,k(x,t)u)_{L^2(Q^\tau)} \\ =\ &(u,f)_{L^2(Q^\tau)} - (\Im_x^2 u,f)_{L^2(Q^\tau)} \\ &+(v,g)_{L^2(Q^\tau)} - (\Im_x^2 v,g)_{L^2(Q^\tau)}. \end{aligned} \tag{19}$$

Using conditions (3)-(8), P_1 and P_2, integrating by parts the twelve terms on the left-hand side of the previous equality, we obtain

$$(u,u_t)_{L^2(Q^\tau)} = \frac{1}{2}\int\int_0^l u^2dx - \frac{1}{2}\int\int_0^l {u_0}^2dx, \tag{20}$$

$$-(u,(a(x,t)u_x)_x)_{L^2(Q^\tau)} = \int\int_{Q^\tau} au_x^2dxdt, \tag{21}$$

$$-(u,(b(x,t)v_x)_x)_{L^2(Q^\tau)} = \int\int_{Q^\tau} bu_xv_xdxdt, \tag{22}$$

$$-(\Im_x^2u,u_t)_{L^2(Q^\tau)} = \frac{1}{2}\int\int_0^l(\Im_xu)^2dx - \frac{1}{2}\int\int_0^l(\Im_xu_0)^2dx, \tag{23}$$

$$(\Im_x^2u,(a(x,t)u_x)_x)_{L^2(Q^\tau)} = \int\int_{Q^\tau} au^2dxdt + \int\int_{Q^\tau} a_xu(\Im_xu)dxdt, \tag{24}$$

$$(\Im_x^2u,(b(x,t)v_x)_x)_{L^2(Q^\tau)} = \int\int_{Q^\tau} buvdxdt + \int\int_{Q^\tau} b_xv(\Im_xu)dxdt, \tag{25}$$

$$(v,v_t)_{L^2(Q^\tau)} = \frac{1}{2}\int\int_0^l v^2dx - \frac{1}{2}\int\int_0^l {v_0}^2dx \tag{26}$$

$$-(v,(c(x,t)v_x)_x)_{L^2(Q^\tau)} = \int\int_{Q^\tau} cv_x^2dxdt, \tag{27}$$

$$-(v,k(x,t)u)_{L^2(Q^\tau)} = -\int\int_{Q^\tau} kuvdxdt, \tag{28}$$

$$-(\Im_x^2v,v_t)_{L^2(Q^\tau)} = \frac{1}{2}\int\int_0^l(\Im_xv)^2dx - \frac{1}{2}\int\int_0^l(\Im_xv_0)^2dx, \tag{29}$$

$$(\Im_x^2v,(c(x,t)v_x)_x)_{L^2(Q^\tau)} = \int\int_{Q^\tau} cv^2dxdt + \int\int_{Q^\tau} c_xv(\Im_xv)dxdt, \tag{30}$$

$$(\Im_x^2v,k(x,t)u)_{L^2(Q^\tau)} = \int\int_{Q^\tau} ku(\Im_x^2v)dxdt. \tag{31}$$

By substitution of (20)-(31) into (19), we obtain

$$\begin{aligned}
&\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Im_xu\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\|\Im_xv\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\sqrt{a}u\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\sqrt{a}u_x\right\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\left\|\sqrt{c}v\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\sqrt{c}v_x\right\|^2_{L^2(\Omega)} \\
=&\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Im_xu_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\|\Im_xv_0\|^2_{L^2(\Omega)} + \int\int_{Q^\tau} ufdxdt - \int\int_{Q^\tau} f(\Im_x^2u)dxdt \\
&+\int\int_{Q^\tau} vgdxdt - \int\int_{Q^\tau} g(\Im_x^2v)dxdt + \int\int_{Q^\tau} buvdxdt \\
&+\int\int_{Q^\tau} kuvdxdt - \int\int_{Q^\tau} a_xu(\Im_xu)dxdt \\
&-\int\int_{Q^\tau} b_xv(\Im_xu)dxdt - \int\int_{Q^\tau} c_xv(\Im_xv)dxdt \\
&-\int\int_{Q^\tau} ku(\Im_x^2v)dxdt - \int\int_{Q^\tau} bu_xv_xdxdt.
\end{aligned} \tag{32}$$

Applying the ε-Cauchy inequality and the conditions P_1 and P_2, to estimate the eleven terms on the right hand side of (32), we obtain

$$\int\int_{Q^\tau} ufdxdt \le \frac{\varepsilon_1}{2}\|u\|^2_{L^2(Q^\tau)} + \frac{1}{2\varepsilon_1}\|f\|^2_{L^2(Q^\tau)}, \tag{33}$$

$$\int\int_{Q^\tau} f(\Im_x^2u)dxdt \le \frac{\varepsilon_2}{2}\|f\|^2_{L^2(Q^\tau)} + \frac{1}{2\varepsilon_2}\left\|\Im_x^2u\right\|^2_{L^2(Q^\tau)}, \tag{34}$$

$$\int\int_{Q^\tau} vgdxdt \le \frac{\varepsilon_3}{2}\|v\|^2_{L^2(Q^\tau)} + \frac{1}{2\varepsilon_3}\|g\|^2_{L^2(Q^\tau)}, \tag{35}$$

$$\int\int_{Q^\tau} g(\Im_x^2v)dxdt \le \frac{\varepsilon_4}{2}\|g\|^2_{L^2(Q^\tau)} + \frac{1}{2\varepsilon_4}\left\|\Im_x^2v\right\|^2_{L^2(Q^\tau)}, \tag{36}$$

$$-\int\int_{Q^\tau} b_1uvdxdt \le \frac{b_1\varepsilon_5}{2}\|u\|^2_{L^2(Q^\tau)} + \frac{b_1}{2\varepsilon_5}\|v\|^2_{L^2(Q^\tau)}, \tag{37}$$

$$-\int\int_{Q^\tau} k_1uvdxdt \le \frac{k_1\varepsilon_6}{2}\|u\|^2_{L^2(Q^\tau)} + \frac{k_1}{2\varepsilon_6}\|v\|^2_{L^2(Q^\tau)}, \tag{38}$$

$$\int\int_{Q^\tau} a_2u(\Im_xu)dxdt \le \frac{a_2\varepsilon_7}{2}\|u\|^2_{L^2(Q^\tau)} + \frac{a_2}{2\varepsilon_7}\|\Im_xu\|^2_{L^2(Q^\tau)}, \tag{39}$$

$$\int\int_{Q^\tau} b_2v(\Im_xu)dxdt \le \frac{b_2\varepsilon_8}{2}\|v\|^2_{L^2(Q^\tau)} + \frac{b_2}{2\varepsilon_8}\|\Im_xu\|^2_{L^2(Q^\tau)}, \tag{40}$$

$$\int\int_{Q^\tau} c_2 v(\Im_x v)dxdt \le \frac{c_2\varepsilon_9}{2}\|v\|^2_{L^2(Q^\tau)} + \frac{c_2}{2\varepsilon_9}\|\Im_x v\|^2_{L^2(Q^\tau)}, \quad (41)$$

$$\int\int_{Q^\tau} k_1 u(\Im_x^2 v)dxdt \le \frac{k_1\varepsilon_{10}}{2}\|u\|^2_{L^2(Q^\tau)} + \frac{k_1}{2\varepsilon_{10}}\|\Im_x^2 v\|^2_{L^2(Q^\tau)}, \quad (42)$$

$$-\int\int_{Q^\tau} b_1 u_x v_x dxdt \le \frac{b_1\varepsilon_{11}}{2}\|u_x\|^2_{L^2(Q^\tau)} + \frac{b_1}{2\varepsilon_{11}}\|v_x\|^2_{L^2(Q^\tau)}. \quad (43)$$

The substitution of (33)-(43) in (32), gives

$$\begin{aligned}
&\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Im_x u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\|\Im_x v\|^2_{L^2(\Omega)} + a_0\|u\|^2_{L^2(Q^\tau)} + a_0\|u_x\|^2_{L^2(Q^\tau)} \\
&+c_0\|v\|^2_{L^2(Q^\tau)} + c_0\|v_x\|^2_{L^2(Q^\tau)} \\
\le\ &\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Im_x u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\|\Im_x v_0\|^2_{L^2(\Omega)} \\
&+\left(\frac{\varepsilon_5 b_1 + (\varepsilon_6+\varepsilon_{10})k_1 + \varepsilon_7 a_2 + \varepsilon_1}{2}\right)\|u\|^2_{L^2(Q^\tau)} \\
&+\left(\frac{\varepsilon_8 b_2 + \varepsilon_9 c_2}{2} + \frac{k_1}{2\varepsilon_6} + \frac{b_1}{2\varepsilon_5}\right)\|v\|^2_{L^2(Q^\tau)} \\
&+\frac{\varepsilon_{11}b_1}{2}\|u_x\|^2_{L^2(Q^\tau)} + \frac{b_1}{2\varepsilon_{11}}\|v_x\|^2_{L^2(Q^\tau)} \\
&+\left(\frac{\varepsilon_2}{2} + \frac{1}{2\varepsilon_2}\right)\|f\|^2_{L^2(Q^\tau)} + \left(\frac{\varepsilon_4}{2} + \frac{1}{2\varepsilon_3}\right)\|g\|^2_{L^2(Q^\tau)} \\
&+\left(\frac{b_2}{2\varepsilon_8} + \frac{a_2}{2\varepsilon_7}\right)\|\Im_x u\|^2_{L^2(Q^\tau)} + \frac{c_2}{2\varepsilon_9}\|\Im_x v\|^2_{L^2(Q^\tau)} \\
&+\frac{1}{2\varepsilon_2}\|\Im_x^2 u\|^2_{L^2(Q^\tau)} + \left(\frac{1}{2\varepsilon_4} + \frac{k_1}{2\varepsilon_{10}}\right)\|\Im_x^2 v\|^2_{L^2(Q^\tau)}.
\end{aligned}$$

If we put $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_8 = \varepsilon_{10}$, $\varepsilon_7 = \frac{2a_0}{a_2}$, $\varepsilon_8 = \frac{2c_0}{a_0}$, $\varepsilon_{11} = \frac{2a_0}{b_1}$, and $b_1^2 - 4a_0c_0 \le 0$, then, we get

$$\begin{aligned}
&\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Im_x u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\|\Im_x v\|^2_{L^2(\Omega)} + a_0\|u\|^2_{L^2(Q^\tau)} + a_0\|u_x\|^2_{L^2(Q^\tau)} \\
&+c_0\|v\|^2_{L^2(Q^\tau)} + c_0\|v_x\|^2_{L^2(Q^\tau)} \\
\le\ &\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Im_x u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\|\Im_x v_0\|^2_{L^2(\Omega)} + \left(\frac{b_1+2k_1+1}{2}\right)\|u\|^2_{L^2(Q^\tau)} \\
&+\left(\frac{b_1+k_1+b_2+1}{2}\right)\|v\|^2_{L^2(Q^\tau)} + \|f\|^2_{L^2(Q^\tau)} \\
&+\|g\|^2_{L^2(Q^\tau)} + \left(\frac{a_2^2}{4a_0} + \frac{b_2}{2}\right)\|\Im_x u\|^2_{L^2(Q^\tau)} \\
&+\frac{c_2^2}{4c_0}\|\Im_x v\|^2_{L^2(Q^\tau)} + \frac{1}{2}\|\Im_x^2 u\|^2_{L^2(Q^\tau)} \\
&+\left(\frac{1+k_1}{2}\right)\|\Im_x^2 v\|^2_{L^2(Q^\tau)}. \qquad (44)
\end{aligned}$$

By using the following inequalities

$$\|\Im_x^m u\|^2_{L^2(\Omega)} \le \frac{l^2}{2}\|\Im_x^{m-1} u\|^2_{L^2(\Omega)},\ \forall m \in IN^*,$$

$$\|\Im_x^m u\|^2_{L^2(Q^\tau)} \le \frac{l^2}{2}\|\Im_x^{m-1} u\|^2_{L^2(Q^\tau)}, \forall m \in IN^*,$$

some right hand side terms of (44) can be estemated as

$$\frac{1}{2}\|\Im_x u_0\|^2_{L^2(\Omega)} \le \frac{l^2}{4}\|u_0\|^2_{L^2(\Omega)}, \quad (45)$$

$$\frac{1}{2}\|\Im_x v_0\|^2_{L^2(\Omega)} \le \frac{l^2}{4}\|v_0\|^2_{L^2(\Omega)}, \quad (46)$$

$$\frac{1}{2}\|\Im_x^2 u\|^2_{L^2(Q^\tau)} \le \frac{l^2}{4}\|\Im_x u\|^2_{L^2(Q^\tau)}, \quad (47)$$

$$\left(\frac{1+k_1}{2}\right)\|\Im_x^2 v\|^2_{L^2(Q^\tau)} \le \frac{(1+k_1)}{4}l^2\|\Im_x v\|^2_{L^2(Q^\tau)}. \quad (48)$$

The substitution of (45)-(48) in (44), gives

$$\begin{aligned}
&\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Im_x u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Im_x v\|^2_{L^2(\Omega)} \\
\le\ &\left(\frac{b_1+2k_1+1}{2}\right)\|u\|^2_{L^2(Q^\tau)} \\
&+\left(\frac{b_1+k_1+b_2+1}{2}\right)\|v\|^2_{L^2(Q^\tau)} \\
&+\left(\frac{a_2^2}{4a_0} + \frac{2b_2+l^2}{4}\right)\|\Im_x u\|^2_{L^2(Q^\tau)} + \|f\|^2_{L^2(Q^\tau)} \\
&+\left(\frac{c_2^2}{4c_0} + \frac{l^2k_1+l^2}{4}\right)\|\Im_x v\|^2_{L^2(Q^\tau)} + \|g\|^2_{L^2(Q^\tau)} \\
&+\left(\frac{2+l^2}{4}\right)\|v_0\|^2_{L^2(\Omega)} + \left(\frac{2+l^2}{4}\right)\|\Im_x v_0\|^2_{L^2(\Omega)}. \qquad (49)
\end{aligned}$$

Then

$$\|u\|^2_{L^2(\Omega)} + \|\Im_x u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|\Im_x v\|^2_{L^2(\Omega)}$$
$$\le K(\|u\|^2_{L^2(Q^\tau)} + \|v\|^2_{L^2(Q^\tau)} + \|\Im_x u\|^2_{L^2(Q^\tau)}$$
$$+ \|\Im_x v\|^2_{L^2(Q^\tau)} + \|f\|^2_{L^2(Q^\tau)} + \|g\|^2_{L^2(Q^\tau)}$$
$$+ \|u_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega)}), \tag{50}$$

where

$$K = \max\{2, b_1 + 2k_1 + 1, b_1 + k_1 + b_2 + 1, \frac{a_2^2}{2a_0} + \frac{2b_2 + l^2}{2}, \frac{c_2^2}{2c_0} + \frac{l^2k_1 + l^2}{2}, \frac{2 + l^2}{2}\}.$$

By applying Gronwall's lemma to (50), we get

$$\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|\Im_x u\|^2_{L^2(\Omega)} + \|\Im_x v\|^2_{L^2(\Omega)}$$
$$\le Ke^{KT}\left[\|f\|^2_{L^2(Q^\tau)} + \|g\|^2_{L^2(Q^\tau)} + \|u_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\right] \tag{51}$$

Taking the supremum with respect to τ over $(0,T)$, we obtain (17) with $C = \sqrt{K}\exp(\frac{KT}{2})$.

Proposition 1. *The operator A acting on $B = B_1 \times B_2$ into $H = H_1 \times H_2$ is closable.*

Proof. Let $\overline{A}$ be the closure of the operator A and $D(A)$ its domain of definition. If $z_n = (u_n, v_n) \in D(A)$ is a sequence such that

$$z_n = (u_n, v_n) \underset{n\to\infty}{\to} (0,0) \text{ into } B, \tag{52}$$

and

$$Az_n = (L_1 u_n, L_2 v_n) \underset{n\to\infty}{\to} F = (F_1, F_2) \text{ into } H, \tag{53}$$

it is necessary to proof that $(F_1, F_2) = (0,0)$, this means

$$f = g = u_0 = v_0. \tag{54}$$

Since

$$(u_n, v_n) \underset{n\to\infty}{\to} (0,0) \text{ into } D'(Q) \times D'(Q), \tag{55}$$

where $D'(Q)$ is the distribution space, then by (51), we obtain

$$(L_1 u_n, L_2 v_n) \underset{n\to\infty}{\to} (0,0) \text{ into } D'(Q) \times D'(Q). \tag{56}$$

But as

$$(\mathscr{L}_1 u_n, \mathscr{L}_2 v_n) \underset{n\to\infty}{\to} (f,g) \text{ into } L^2(Q) \times L^2(Q), \tag{57}$$

then

$$(\mathscr{L}_1 u_n, \mathscr{L}_2 v_n) \underset{n\to\infty}{\to} (f,g) \text{ into } D'(Q) \times D'(Q). \tag{58}$$

By virtue of the uniqueness of the limit into $D'(Q) \times D'(Q)$, we conclude from (56)-(58) that

$$(f,g) = (0,0). \tag{59}$$

From (53), we get

$$l_1 u_n \underset{n\to\infty}{\to} u_0 \text{ into } L^2(\Omega). \tag{60}$$

Then, we deduce that

$$l_1 u_n \underset{n\to\infty}{\to} u_0 \text{ into } D'(\Omega). \tag{61}$$

On the other hand as

$$z_n \underset{n\to\infty}{\to} 0 \text{ into } B, \tag{62}$$

and

$$\|l_1 u_n\|^2_{L^2(\Omega)} \le \|z_n\|^2_B, \forall n \in IN, \tag{63}$$

then, we get

$$l_1 u_n \underset{n\to\infty}{\to} 0 \text{ into } L^2(\Omega). \tag{64}$$

Consequently

$$l_1 u_n \underset{n\to\infty}{\to} 0 \text{ into } D'(\Omega). \tag{65}$$

Then, by virtue of the uniqueness of the limit in $D'(\Omega)$, we obtain

$$u_0 = 0,$$

and in the same way, we proof that

$$v_0 = 0.$$

Then A is closed.

The solution of the operator equation

$$\overline{A}z = \mathscr{F},$$

is called strong solution of problem (1)-(8). We extend inequality (17) to the set of solutions $u \in D(\overline{A})$ by passing to the limit and thus establish uniqueness of a strong solution and closedness of the range $R(A)$ of the operator A in the space H.
Then, there exists a positive constant C such that

$$\|z\|_B \le C\left\|\overline{A}z\right\|_H, \qquad \forall z \in D(\overline{A}).$$

We now obtain two corollaries.

Corollary 1. *The strong solution of problem (1)-(8) if exists is unique, and dependent with $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \in H$, where $\mathscr{F}_1 = (f, u_0)$ and $\mathscr{F}_2 = (g, v_0)$.*

Corollary 2. *The range $R(\overline{A})$ of the operator $\overline{A}$ is closed in the space H, and $R(\overline{A}) = \overline{R(A)}$.*

5 Existence of solution

Theorem 2. *For all $(f,g) \in (L^2(Q))^2$ and $(u_0,v_0) \in (L^2(Q))^2$ there exists a unique strong solution*

$$z = \overline{A}^{-1}\mathscr{F} = \overline{A^{-1}}\mathscr{F},$$

of problem (1)-(8), with $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \in H$, $\mathscr{F}_1 = (f,u_0)$, $\mathscr{F}_2 = (g,v_0), z = (u,v)$, and

$$\|z\|_B \le C\|Az\|_H, \qquad \forall z \in D(\overline{A}),$$

where C is a positive constant.

Proof. It is enough to proof that the range of A is dense in $H = H_1 \times H_2$. Let $\Psi = (\Psi_1, \Psi_2) = (\{w_1,w_3\},\{w_2,w_4\}) \in R(A)^{\perp}$, such as

$$\begin{aligned}(Az,\Psi)_H &= (\{L_1u, L_2v\}, (\Psi_1,\Psi_2))_H \\ &= (\mathscr{L}_1u, w_1)_{L^2(Q)} + (l_1u, w_3)_{L^2(\Omega)} \\ &\quad + (\mathscr{L}_2v, w_2)_{L^2(Q)} + (l_2v, w_4)_{L^2(\Omega)} \\ &= 0.\end{aligned}$$

For $z \in D_0(A)$, we get

$$(\mathscr{L}_1u, w_1)_{L^2(Q)} + (\mathscr{L}_2v, w_2)_{L^2(Q)} = 0,$$

then

$$(w_1,w_2) = (0,0).$$

Proposition 2. *If for all function $W = (w_1,w_2) \in (L^2(Q))^2$ and for all function*

$$z \in D_0(A) = \{z \quad / z \in D(A) : \ell_1 u = \ell_2 v = 0\},$$

we have

$$(\mathscr{L}_1u, w_1)_{L^2(Q)} + (\mathscr{L}_2v, w_2)_{L^2(Q)} = 0, \tag{66}$$

then $W = 0$ almost everywhere in Q.

Proof. It follows from (66) holds for any $z \in D_0(A)$. We can express it in a particular form, we put

$$z = (u,v) = \begin{cases} (0,0) & 0 \le t \le s, \\ (\int\int_s^t u_\tau d\tau, \int\int_s^t v_\tau d\tau) & s \le t \le T, \end{cases} \tag{67}$$

and let $h_i(x,t), i = \overline{1,2}$, defined by

$$\begin{cases} h_1(x,t) = -u_t - \Im_x^2 u, \\ h_2(x,t) = -v_t - \Im_x^2 v. \end{cases} \tag{68}$$

It's easy to see that

$$W = (w_1,w_2) = \begin{cases} w_1 = -u_{tt} - \Im_x^2 u_t, \\ w_2 = -v_{tt} - \Im_x^2 v_t. \end{cases} \tag{69}$$

In accordance with (67) et (68), the function $z = (u,v) \in (L^2(Q))^2$. In fact z possessed a superior order of regularity.

Lemma 1. *The function $W = (w_1,w_2)$ defined by (69) is into $(L^2(Q))^2$.*

Proof. We use the t-operators of regularization

$$(\rho_\varepsilon f)(x,t) = \frac{1}{\varepsilon}\int\int_0^T w(\frac{t-s}{\varepsilon}) f(x,s)ds,$$

where w$\in C^\infty(0,t), w = 0$ at the neighborhoods $t = 0, t = T$ and outside $(0,t)$, *and* $\int\int_{IR} w(s)ds = 1$.
By applying $\frac{\partial}{\partial t}$ to the first equation of (68), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(-u_t - \Im_x^2 u\right) &= \frac{\partial}{\partial t}\left[\left(-u_t - \Im_x^2 u\right) - \rho_\varepsilon\left(-u_t - \Im_x^2 u\right)\right] \\ &\quad + \frac{\partial}{\partial t}\left(\rho_\varepsilon(h_1)\right).\end{aligned}$$

Then

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}\left(-u_t - \Im_x^2 u\right)\right\|^2_{L^2(Q^\tau)} \\ &\le 2\left\|\frac{\partial}{\partial t}\left[\left(-u_t - \Im_x^2 u\right) - \rho_\varepsilon\left(-u_t - \Im_x^2 u\right)\right]\right\|^2_{L^2(Q^\tau)} \\ &\quad + 2\left\|\frac{\partial}{\partial t}\left(\rho_\varepsilon(h_1)\right)\right\|^2_{L^2(Q^\tau)}.\end{aligned}$$

As

$$\underset{\varepsilon \to 0}{\left\|\frac{\partial}{\partial t}\left[\left(-u_t - \Im_x^2 u\right) - \rho_\varepsilon\left(-u_t - \Im_x^2 u\right)\right]\right\|^2_{L^2(Q^\tau)} \to 0,},$$

we get

$$\left\|\frac{\partial}{\partial t}\left(-u_t - \Im_x^2 u\right)\right\|^2_{L^2(Q^\tau)} \le 2\left\|\frac{\partial}{\partial t}\left(\rho_\varepsilon(h_1)\right)\right\|^2_{L^2(Q^\tau)}.$$

As $(\rho_\varepsilon(h_1)) \to h_1$ and $\frac{\partial}{\partial t}\left(-u_t - \Im_x^2 u\right)$ is bounded in $L^2(Q)$, then $w_1 \in L^2(Q)$. With the same methode, we proof that $w_2 \in L^2(Q)$.

For continue the proof of proposition, substituting W into (66) with representation given by (69), we obtain

$$(-u_{tt} - \Im_x^2 u_t, \mathscr{L}_1 u)_{L^2(Q)} + (-v_{tt} - \Im_x^2 v, \mathscr{L}_2 v_t,)_{L^2(Q)} = 0.$$

Then

$$\begin{aligned}&-(u_{tt}, u_t)_{L^2(Q)} + (u_{tt}, (a(x,t)u_x)_x)_{L^2(Q)} \\ &+(u_{tt}, (b(x,t)v_x)_x)_{L^2(Q)} - (\Im_x^2 u_t, u_t)_{L^2(Q)} \\ &+(\Im_x^2 u_t, (a(x,t)u_x)_x)_{L^2(Q)} + (\Im_x^2 u_t, (b(x,t)v_x)_x)_{L^2(Q)} \\ &-(v_{tt}, v_t)_{L^2(Q)} + (v_{tt}, (c(x,t)v_x)_x)_{L^2(Q)} \\ &+(v_{tt}, k(x,t)u)_{L^2(Q)} - (\Im_x^2 v_t, v_t)_{L^2(Q)} \\ &+(\Im_x^2 v_t, (c(x,t)v_x)_x)_{L^2(Q)} + (\Im_x^2 v_t, k(x,t)u)_{L^2(Q)} \\ &= 0.\end{aligned} \tag{70}$$

By integration by parts and using conditions (3)-(8), we obtain

$$-(u_{tt},u_t)_{L^2(Q^s)} = \frac{1}{2}\|u_t(x,s)\|^2_{L^2(\Omega)}, \quad (71)$$

$$a(u_{tt},(au_x)_x)_{L^2(Q^s)} = \left\|\sqrt{a}u_{xt}(x,s)\right\|^2_{L^2(Q^s)} + \frac{1}{2}\|\sqrt{a_t}u_x(x,T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\sqrt{a_{tt}}u_x(x,s)\|^2_{L^2(Q^s)}, \quad (72)$$

$$(u_{tt},(b(x,t)v_x)_x)_{L^2(Q^s)} = \int\int_{Q^s} bu_{xt}v_{xt}dxdt + \int\int_{Q^s} b_tu_{xt}v_xdxdt, \quad (73)$$

$$-(\Im_x^2u_t,u_t)_{L^2(Q^s)} = \|\Im_xu_t\|^2_{L^2(Q^s)}, \quad (74)$$

$$(\Im_x^2u_t,(a(x,t)u_x)_x)_{L^2(Q^s)} = \int\int_{Q^s} a_xu(\Im_xu_t)dxdt + \frac{1}{2}\left\|\sqrt{a}u(x,T)\right\|^2_{L^2(\Omega)} - \frac{1}{2}\|\sqrt{a_t}u(x,s)\|^2_{L^2(Q^s)}, \quad (75)$$

$$(\Im_x^2u_t,(b(x,t)v_x)_x)_{L^2(Q^s)} = -\int\int_{Q^s} bv_x(\Im_xu_t)dxdt, \quad (76)$$

$$-(v_{tt},v_t)_{L^2(Q^s)} = \frac{1}{2}\|v_t(x,s)\|^2_{L^2(\Omega)}, \quad (77)$$

$$(v_{tt},(c(x,t)v_x)_x)_{L^2(Q^s)} = \left\|\sqrt{c}v_{xt}(x,s)\right\|^2_{L^2(Q^s)} + \frac{1}{2}\|\sqrt{c_t}v_x(x,T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\sqrt{c_{tt}}v_x(x,s)\|^2_{L^2(Q^s)}, \quad (78)$$

$$(v_{tt},k(x,t)u)_{L^2(Q^s)} = -\int\int_{Q^s} k_tuv_tdxdt - \int\int_{Q^s} ku_tv_tdxdt, \quad (79)$$

$$-(\Im_x^2v_t,v_t)_{L^2(Q)} = \|\Im_xv_t\|^2_{L^2(Q^s)}, \quad (80)$$

$$(\Im_x^2v_t,(c(x,t)v_x)_x)_{L^2(Q)} = \int\int_{Q^s} c_xv(\Im_xv_t)dxdt + \frac{1}{2}\left\|\sqrt{c}v(x,T)\right\|^2_{L^2(\Omega)} - \frac{1}{2}\|\sqrt{c_t}v(x,s)\|^2_{L^2(Q^s)}, \quad (81)$$

$$-(\Im_x^2v_t,k(x,t)u)_{L^2(Q)} = \int\int_{Q^s} ku(\Im_x^2v_t)dxdt. \quad (82)$$

By substitution of (71)-(82) in (70), we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\sqrt{a}u(x,T)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sqrt{a_t}u_x(x,T)\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\|u_t(x,s)\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\sqrt{c}v(x,T)\right\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\|\sqrt{c_t}v_x(x,T)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_t(x,s)\|^2_{L^2(\Omega)} \\
&+\|\Im_xu_t\|^2_{L^2(Q^s)} + \|\Im_xv_t\|^2_{L^2(Q^s)} \\
&+\left\|\sqrt{a}u_{xt}(x,s)\right\|^2_{L^2(Q^s)} + \left\|\sqrt{c}v_{xt}(x,s)\right\|^2_{L^2(Q^s)} \\
=&\ \frac{1}{2}\|\sqrt{a_t}u(x,s)\|^2_{L^2(Q^s)} + \frac{1}{2}\|\sqrt{a_{tt}}u_x(x,s)\|^2_{L^2(Q^s)} \\
&+\frac{1}{2}\|\sqrt{c_t}v(x,s)\|^2_{L^2(Q^s)} + \frac{1}{2}\|\sqrt{c_{tt}}v_x(x,s)\|^2_{L^2(Q^s)} \\
&-\int\int_{Q^s} a_xu(\Im_xu_t)dxdt + \int\int_{Q^s} bv_x(\Im_xu_t)dxdt \\
&+\int\int_{Q^s} c_xv(\Im_xv_t)dxdt + \int\int_{Q^s} k_tuv_tdxdt \\
&+\int\int_{Q^s} ku_tv_tdxdt - \int\int_{Q^s} ku(\Im_x^2v_t)dxdt \\
&-\int\int_{Q^s} bu_{xt}v_{xt}dxdt - \int\int_{Q^s} b_tu_{xt}v_xdxdt. \quad (83)
\end{aligned}$$

Using ε-Cauchy inequality and conditions P_1 and P_2, we obtain

$$-\int\int_{Q^s} a_xu(\Im_xu_t)dxdt \le \frac{a_2\varepsilon_1}{2}\|u\|^2_{L^2(Q^s)} + \frac{a_2}{2\varepsilon_1}\|(\Im_xu_t)\|^2_{L^2(Q^s)}, \quad (84)$$

$$\int\int_{Q^s} bv_x(\Im_xu_t)dxdt \le \frac{b_1\varepsilon_2}{2}\|v_x\|^2_{L^2(Q^s)} + \frac{b_1}{2\varepsilon_2}\|(\Im_xu_t)\|^2_{L^2(Q^s)}, \quad (85)$$

$$\int\int_{Q^s} c_xv(\Im_xv_t)dxdt \le \frac{c_2\varepsilon_3}{2}\|v\|^2_{L^2(Q^s)} + \frac{c_2}{2\varepsilon_3}\|(\Im_xv_t)\|^2_{L^2(Q^s)}, \quad (86)$$

$$\int\int_{Q^s} k_tuv_tdxdt \le \frac{k_2\varepsilon_4}{2}\|u\|^2_{L^2(Q^s)} + \frac{k_2}{2\varepsilon_4}\|v_t\|^2_{L^2(Q^s)}, \quad (87)$$

$$\int\int_{Q^s} ku_tv_tdxdt \le \frac{k_1\varepsilon_5}{2}\|u_t\|^2_{L^2(Q^s)} + \frac{k_1}{2\varepsilon_5}\|v_t\|^2_{L^2(Q^s)}, \quad (88)$$

$$-\int\int_{Q^s} ku(\Im_x^2v_t)dxdt \le \frac{k_1\varepsilon_6}{2}\|u\|^2_{L^2(Q^s)} + \frac{k_1}{2\varepsilon_6}\left\|(\Im_x^2v_t)\right\|^2_{L^2(Q^s)}, \quad (89)$$

$$-\int\int_{Q^s} bu_{xt}v_{xt}dxdt-\int\int_{Q^S} b_t u_{xt}v_x dxdt$$
$$=-\int\int_{Q^s} u_{xt}\left(bv_{xt}+b_t v_x\right)dxdt$$
$$\leq \frac{\varepsilon_7}{2}\|u_{xt}\|^2_{L^2(Q^s)}+\frac{1}{2\varepsilon_7}\int\int_{Q^s}\left(\frac{\partial}{\partial t}(bv_x)\right)^2 dxdt$$
$$\leq \frac{\varepsilon_7}{2}\|u_{xt}\|^2_{L^2(Q^s)}+\frac{b_1^2}{2\varepsilon_7}\|v_{xt}\|^2_{L^2(Q^s)}. \tag{90}$$

By substitution of (84)-(90) in (83), we obtain

$$\begin{aligned}
&\frac{a_0}{2}\|u(x,T)\|^2_{L^2(\Omega)}+\frac{a_3}{2}\|u_x(x,T)\|^2_{L^2(\Omega)}\\
&+\frac{1}{2}\|u_t(x,s)\|^2_{L^2(\Omega)}+\frac{c_0}{2}\|v(x,T)\|^2_{L^2(\Omega)}\\
&+\frac{c_3}{2}\|v_x(x,T)\|^2_{L^2(\Omega)}+\frac{1}{2}\|v_t(x,s)\|^2_{L^2(\Omega)}\\
&+\|\Im_x u_t\|^2_{L^2(Q^s)}+\|\Im_x v_t\|^2_{L^2(Q^s)}\\
&+\frac{a_0}{2}\|u_{xt}(x,s)\|^2_{L^2(Q^s)}+\frac{c_0}{2}\|v_{xt}(x,s)\|^2_{L^2(Q^s)}\\
\leq\; &\frac{(a_1\varepsilon_1+k_2\varepsilon_4+k_1\varepsilon_6)}{2}\|u\|^2_{L^2(Q^s)}+\frac{a_5}{2}\|u_x\|^2_{L^2(Q^s)}\\
&+\frac{k_1\varepsilon_5}{2}\|u_t\|^2_{L^2(Q^s)}+\frac{(c_2\varepsilon_3+c_4)}{2}\|v\|^2_{L^2(Q^s)}\\
&+\frac{(b_1\varepsilon_2+c_5)}{2}\|v_x\|^2_{L^2(Q^s)}+\left(\frac{k_2}{2\varepsilon_4}+\frac{k_1}{2\varepsilon_5}\right)\|v_t\|^2_{L^2(Q^s)}\\
&+\left(\frac{a_2}{2\varepsilon_1}+\frac{b_1}{2\varepsilon_2}\right)\|\Im_x u_t\|^2_{L^2(Q^s)}\\
&+\left(\frac{c_2}{2\varepsilon_3}+\frac{k_1}{2\varepsilon_6}\right)\|\Im_x v_t\|^2_{L^2(Q^s)}+\frac{k_1}{2\varepsilon_6}\left\|(\Im_x^2 v_t)\right\|^2_{L^2(Q^s)}\\
&+\frac{\varepsilon_7}{2}\|u_{xt}\|^2_{L^2(Q^s)}+\frac{b_1^2}{2\varepsilon_7}\|v_{xt}\|^2_{L^2(Q^s)}.
\end{aligned}\tag{91}$$

If we put $\varepsilon_1=\frac{a_2}{2},\varepsilon_2=\varepsilon_4=\varepsilon_5=\varepsilon_6=1,\ \varepsilon_7=a_0,\varepsilon_3=\frac{c_2}{2}$ and $b_1^2-a_0c_0\leq 0$, then form (91), we obtain

$$\begin{aligned}
&\frac{a_0}{2}\|u(x,T)\|^2_{L^2(\Omega)}+\frac{a_3}{2}\|u_x(x,T)\|^2_{L^2(\Omega)}\\
&+\frac{1}{2}\|u_t(x,s)\|^2_{L^2(\Omega)}+\frac{c_0}{2}\|v(x,T)\|^2_{L^2(\Omega)}\\
&+\frac{c_3}{2}\|v_x(x,T)\|^2_{L^2(\Omega)}+\frac{1}{2}\|v_t(x,s)\|^2_{L^2(\Omega)}\\
\leq\; &\frac{(a_1^2+2k_2+2k_1)}{4}\|u\|^2_{L^2(Q^s)}+\frac{1}{2}\|u_x\|^2_{L^2(Q^s)}+\frac{k_1}{2}\|u_t\|^2_{L^2(Q^s)}\\
&+\frac{(c_2^2+2c_4)}{2}\|v\|^2_{L^2(Q^s)}+\frac{(b_1+c_5)}{2}\|v_x\|^2_{L^2(Q^s)}\\
&+\left(\frac{k_1+k_2}{2}\right)\|v_t\|^2_{L^2(Q^s)}+\frac{b_1}{2}\|\Im_x u_t\|^2_{L^2(Q^s)}\\
&+\frac{k_1}{2}\|\Im_x v_t\|^2_{L^2(Q^s)}+\frac{k_1}{2}\|\Im_x^2 v_t\|^2_{L^2(Q^s)}.
\end{aligned}\tag{92}$$

Using this inequality

$$\|\Im_x^m u\|^2_{L^2(Q^s)}\leq\frac{l^2}{2}\left\|\Im_x^{m-1}u\right\|^2_{L^2(Q^s)},\forall m\in IN^*,$$

we have

$$\frac{b_1}{2}\|\Im_x u_t\|^2_{L^2(Q^s)}\leq\frac{b_1 l^2}{4}\|u_t\|^2_{L^2(Q^s)}, \tag{93}$$
$$\frac{k_1}{2}\|\Im_x v_t\|^2_{L^2(Q^s)}\leq\frac{k_1 l^2}{4}\|v_t\|^2_{L^2(Q^s)}, \tag{94}$$
$$\frac{k_1}{2}\|\Im_x^2 v_t\|^2_{L^2(Q^s)}\leq\frac{k_1 l^4}{8}\|v_t\|^2_{L^2(Q^s)}. \tag{95}$$

By the substitution of (93)-(95) in (92), we obtain

$$\begin{aligned}
&\frac{a_0}{2}\|u(x,T)\|^2_{L^2(\Omega)}+\frac{a_3}{2}\|u_x(x,T)\|^2_{L^2(\Omega)}\\
&+\frac{1}{2}\|u_t(x,s)\|^2_{L^2(\Omega)}+\frac{c_0}{2}\|v(x,T)\|^2_{L^2(\Omega)}\\
&+\frac{c_3}{2}\|v_x(x,T)\|^2_{L^2(\Omega)}+\frac{1}{2}\|v_t(x,s)\|^2_{L^2(\Omega)}\\
\leq\; &\left(\frac{a_1^2+2k_2+2k_1}{4}\right)\|u\|^2_{L^2(Q^s)}+\frac{1}{2}\|u_x\|^2_{L^2(Q^s)}\\
&+\left(\frac{2k_1+b_1l^2}{4}\right)\|u_t\|^2_{L^2(Q^s)}+\left(\frac{c_2^2+2c_4}{4}\right)\|v\|^2_{L^2(Q^s)}\\
&+\left(\frac{b_1+c_5}{2}\right)\|v_x\|^2_{L^2(Q^s)}+\frac{k_3}{8}\|v_t\|^2_{L^2(Q^s)},
\end{aligned}$$

where

$$k_3=4k_2+(4+l^2+l^4)k_1.$$

Consequently, we get

$$\begin{aligned}
&\|u(x,T)\|^2_{L^2(\Omega)}+\|u_x(x,T)\|^2_{L^2(\Omega)}+\|u_t(x,s)\|^2_{L^2(\Omega)}\\
&\|v(x,T)\|^2_{L^2(\Omega)}+\|v_x(x,T)\|^2_{L^2(\Omega)}+\|v_t(x,s)\|^2_{L^2(\Omega)}\\
\leq\; &K\Big(\|u\|^2_{L^2(Q^s)}+\|u_x\|^2_{L^2(Q^s)}+\|u_t\|^2_{L^2(Q^s)}+\|v\|^2_{L^2(Q^s)}\\
&+\|v_x\|^2_{L^2(Q^s)}+\|v_t\|^2_{L^2(Q^s)}\Big),
\end{aligned}\tag{96}$$

where

$$K=\frac{\max(1,\frac{a_1^2+2k_2+2k_1}{2},\frac{2k_1+b_1l^2}{2},\frac{c_2^2+2c_4}{2},b_1+c_5,\frac{k_3}{4})}{\min(1,a_0,a_3,c_0,c_3)}.$$

We will introduce the function V defined by

$$V=(\alpha(x,t),\beta(x,t))=(\int\int_t^T u_\tau d\tau,\int\int_t^T v_\tau d\tau),$$

where

$$\begin{array}{ll} u(x,t)=\alpha(x,s)-\alpha(x,t), & u(x,T)=\alpha(x,s),\\ v(x,t)=\beta(x,s)-\beta(x,t), & v(x,T)=\beta(x,s).\end{array}$$

From the combination of (96) and (75), we obtain

$$\begin{aligned}
&\|u_t(x,s)\|^2_{L^2(\Omega)}+\|v_t(x,s)\|^2_{L^2(\Omega)}\\
&+(1-2K(T-s))\Big(\|\alpha(x,s)\|^2_{L^2(\Omega)}+\|\alpha_x(x,s)\|^2_{L^2(\Omega)}\\
&+\|\beta(x,s)\|^2_{L^2(\Omega)}+\|\beta_x(x,s)\|^2_{L^2(\Omega)}\Big)\\
\leq\; &2K\Big(\|u_t\|^2_{L^2(Q^s)}+\|v_t\|^2_{L^2(Q^s)}+\|\alpha\|^2_{L^2(Q^s)}\\
&+\|\alpha_x\|^2_{L^2(Q^s)}+\|\beta\|^2_{L^2(Q^s)}+\|\beta_x\|^2_{L^2(Q^s)}\Big).
\end{aligned}\tag{97}$$

If for $s_0 \geq 0 : (1-2K(T-s_0)) = \frac{1}{2}$, then from (97), we get

$$\|u_t(x,s)\|^2_{L^2(\Omega)} + \|v_t(x,s)\|^2_{L^2(\Omega)} + \Big(\|\alpha(x,s)\|^2_{L^2(\Omega)} + \|\alpha_x(x,s)\|^2_{L^2(\Omega)} + \|\beta(x,s)\|^2_{L^2(\Omega)} + \|\beta_x(x,s)\|^2_{L^2(\Omega)}\Big) \leq 4K\Big(\|u_t\|^2_{L^2(Q^s)} + \|v_t\|^2_{L^2(Q^s)} + \|\alpha\|^2_{L^2(Q^s)} + \|\alpha_x\|^2_{L^2(Q^s)} + \|\beta\|^2_{L^2(Q^s)} + \|\beta_x\|^2_{L^2(Q^s)}\Big). \quad (98)$$

We take

$$h(s) = \int_s^T (\|u_t(x,s)\|^2_{L^2(\Omega)} + \|v_t(x,s)\|^2_{L^2(\Omega)} + \|\alpha(x,s)\|^2_{L^2(\Omega)} + \|\alpha_x(x,s)\|^2_{L^2(\Omega)} + \|\beta(x,s)\|^2_{L^2(\Omega)} + \|\beta_x(x,s)\|^2_{L^2(\Omega)})dt.$$

Then from (98), we obtain

$$-h'(s) \leq 4Kh(s),$$

and consequently, we have

$$h(s)\exp 4Ks \leq 0.$$

Then, we obtain

$$h(s) = 0,\ \forall s \in [T-s_0, T].$$

We deduce that $(Az, \Psi)_H = 0$ and then $W = (w_1, w_2) = (0,0)$ into Q_{T-s_0}. Proceeding the same way step by step a finite number of times, we proof that $W = 0$ into Q. Then $w_1 = w_2 = 0$.
The equality $(Az, \Psi)_H = 0$, implies that

$$(\ell_1 u, w_3)_{L^2(\Omega)} + (\ell_2 v, w_4)_{L^2(\Omega)} = 0, for\ all\ z \in D(A).$$

Since the quantities $\ell_1 u$ and $\ell_2 v$ are independent and the ranges of the trace operators ℓ_1, ℓ_2 are dense in $L^2(\Omega)$, then $w_3 = w_4 = 0$. Consequently $R(L)^{\perp} = \{0\} \Leftrightarrow \overline{R(A)} = H$. This completes the proof of Theorem.

6 Conclusion

In the present study, a mixed problem with a classical Neumann boundary condition and a nonlocal integral condition for a parabolic system is investigated. The existence and uniqueness of a strong solution for this problem are established.

The method used for treating this problem is based on the energy inequalities method which is considered as an effective method for the study of linear problems.

We encountered some difficulties during this study, as for example:

– The choice of the multiplier for establishing a priori estimates for the solutions;
– The choice of a suitable function space for our study.

While our work is completed, but there are still some open questions and suggestions for future research, we quote as an example:

– It would be interesting to apply this method for systems of higher orders and of several variables;
– It would be interesting to apply this method for semilinear, quasi-linear and non-linear equations, in one-dimensional or multidimensional structures.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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