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The Solution of Quantum Kinetic Equation with Delta Potential and its Application for Information Technology

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Abstract: The existence of unique solution, in terms of initial data of the hierarchy of quantum kinetic equations with delta potential and application of kinetic equation for information technology, has been proven. The proof is based on the nonrelativistic quantum mechanics and application of semigroup theory methods

Keywords: delta potential, Bethe ansatz, chain BBGKY of quantum kinetic equations

1 Introduction

Bethe Anzats was first introduced in 1931 by Hans Bethe [1] when he considered the Heisenberg model. Since then, this model remains one of the several exactly solvable models of many-body physics [1]-[4], [8]-[17]. A significant contribution was made by the work of Lieb-Liniger [2], in the development of this model, where one-dimensional quantum system of bosons interacting through a potential in the form of a delta function

$$\delta(x-x_0)=\{^{\infty,\quad if\quad x=x_0,}_{0\quad if\quad x\neq x_0}$$

with a Hamiltonian:

$$\mathsf{H}=-\sum_{i=0}^{N}\frac{\partial^2}{\partial x_i^2}+2c\sum_{i<j}\delta(x_i-x_j) \tag{1}$$

is considered.

In (1) x is one dimentional coordinate of a particle, the constant $\hbar$ is called the reduced Planck constant, m is the mass and $\delta(x_i-x_j)$ is the potential energy, $c>0$ (the repulsive case) and $\mathscr{R}:\quad all\quad 0\leq x_i\leq L, i=1,...,N$, where N-the number of particles and L-the size of the periodic box. Here we suppose that $\hbar=2m=1$.

In [2] the of solving of the differential problem related to the Schrödinger equation in $\mathscr{R}:0\leq x_1\leq x_2\leq ...\leq x_N\leq L$ is reduced to a solution of a much simpler system of algebraic equations, called the Bethe ansatz equations.

$$\psi|_{x_j=x_{k+0}}=\psi|_{x_j=x_{k-0}},$$

$$\left(\frac{\partial\psi}{\partial x_j}-\frac{\partial\psi}{\partial x_k}\right)|_{x_j=x_{k+0}}-\left(\frac{\partial\psi}{\partial x_j}-\frac{\partial\psi}{\partial x_k}\right)|_{x_j=x_{k-0}}=2c\psi|_{x_j=x_k},$$

for all $x_j=x_k$ for all j,k=1,2,...,N and $j\neq k$. The solution of the Schrödinger equation in $\mathscr{R}:0\leq x_1\leq x_2\leq ...\leq x_N\leq L$ in this case will have Bethe anzats form:

$$\psi_B(x_1,...x_N)=\sum_{\sigma}\mathsf{A}(\sigma)\sigma exp(i\sum_{j=1}^{N}k_jx_j),$$

where the summation extends over all permutations σ of ordered N numbers $k=k_1,...,k_N$, and $\mathsf{A}(\sigma)=\prod\{S_{\alpha\beta}:\alpha\beta$ is an inversion in $\sigma\}$, are certain coefficients depending on σ:

$$S_{\alpha\beta}=-\frac{c-i(k_\alpha-k_\beta)}{c+i(k_\alpha-k_\beta)}=-exp(i\theta_{\alpha,\beta}),$$

where $\theta_{\alpha,\beta}=\theta(k_\alpha-k_\beta)$, $\theta(r)=-2tan^{-}1(\frac{r}{c})$. Assuming r to be real we have $\pi\geq\theta(r)\geq-\pi$;

In [3] using the ideas of Bethe Ansatz, the method is given to solve the time-dependent Schrödinger equation for a system of one-dimentional bosons interacting via the repulsive delta function potential. Authors of [3] solve time-dependent Schrödinger equation:

$$\mathsf{H}\psi=i\frac{\partial\psi}{\partial t},$$

with initial condition

$$\psi(x;0)=\psi(x_1,x_2,...,x_N;0)=\psi_0(x_1,x_2,...,x_N).$$

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The solution is reduced to

$$i\frac{\partial \psi_\delta}{\partial t} = -\sum_i \frac{\partial^2 \psi_\delta}{\partial x_i^2}$$

in the interior of $\mathscr{R}$, e.g. $\mathscr{R}^0 : x_1 < x_2, \ldots < x_N$ with the initial condition for the Schrödinger equation

$$\psi_\delta(x;0) = \prod_{j=1}^{N} \delta(x_j - y_j) \tag{2}$$

in $\mathscr{R} : -\infty < x_1 < x_2, \ldots < x_N < \infty$ and the boundary condition:

$$\left(\frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j}\right)\psi_\delta|_{x_{j+1}=x_j} = c\psi_\delta|_{x_{j+1}=x_j}. \tag{3}$$

for a system of one-dimensional bosons interacting via the delta function potential. In (2) $y_j \in \mathbb{R}$ are fixed and $y_1 < y_2, \ldots < y_N$. The equation (3) is the effect of the δ-function confined to the boundary of $\mathscr{R}$, e.g. on the hyperplanes $x_{j+1} = x_j$. The interest of authors [3] in the Lieb-Liniger model has arisen because of its connection to ultracold gases confined in a quasi one-dimensional trap [4].

2 The dynamics of a one-dimensional system of Bosons interacting via the delta function potential

In paper we suggest the approach to solve the time-dependent Bogolubov-Born-Green-Kirkwood-Yvon (BBGKY) chain of quantum kinetic equations [5],[6] for a one-dimensional system of bosons interacting via the delta function potential.

We will consider a system of N particles contained in a finite region L. The operators ρ_N^L and Hamiltonian H_N^L act in the space $\mathscr{H}$ with zero boundary condition [7]. Finally we get the equation

$$i\frac{\partial \rho_s^L(x_1,\ldots,x_s;x_1',\ldots,x_s',t)}{\partial t} = [\mathsf{H}_s^L,\rho_s^L](x_1,\ldots,x_s;x_1',\ldots,x_s',t)$$

$$+\frac{N}{L}\left(1-\frac{s}{N}\right)Tr_{x_{s+1}}\sum_{1\le i\le s}(\phi_{i,s+1}(|x_i - x_{s+1}|)-$$

$$\phi_{i,s+1}(|x_i' - x_{s+1}|))\rho_{s+1}^L(x_1,\ldots,x_s,x_{s+1};x_1',\ldots,x_s',x_{s+1},t), \tag{4}$$

with the initial condition

$$\rho_s^L(x_1,\ldots,x_s;x_1',\ldots,x_s',t)|_{t=0} = \rho_s^L(x_1,\ldots,x_s;x_1',\ldots,x_s',0). \tag{5}$$

for $1 \le s < N$. For $s = N$, we have

$$i\frac{\partial \rho_s^L(x_1,\ldots,x_s;x_1',\ldots,x_s',t)}{\partial t} =$$

$$[\mathsf{H}_s^L,\rho_s^L](x_1,\ldots,x_s;x_1',\ldots,x_s',t)].$$

In the problem given by equation (4) and (5) x_i gives the position of ith particle in the 1-dimensional space $\mathscr{R}$, x_i, $i = 1,2,\ldots,s$. In (1) $\hbar = 1$ is the reduced Planck constant and $[,]$ denotes the Poisson bracket.

The reduced statistical operator of s particles is $\rho_s^L(x_1,..,x_s;x_1',..,x_s')$ related to the positive symmetric density matrix D_N^L of N particles by [5],[6]

$$\rho_s^L(x_1,.,x_s;x_1',.,x_s') =$$

$$L^s Tr_{x_{s+1},..,x_N} D_N^L(x_1,.,x_s,x_{s+1},.,x_N;x_1',.,x_s',x_{s+1},.,x_N),$$

where $s \in N$, N is the number of particles, L the size of the periodic box. The trace is defined in terms of the kernel $\rho^L(x,x')$ by the formula

$$Tr_x\rho^L = \int_L \rho^L(x,x)dx.$$

The Hamiltonian of system is defined as

$$\mathsf{H}_s^L(x_1,\ldots,x_s) = \sum_{1\le i\le s}\left(-\frac{1}{2m}\triangle_{x_i} + u^L(x_i)\right) +$$

$$\sum_{1\le i<j\le s}\phi_{i,j}(|x_i - x_j|),$$

where $2m = 1$, $\triangle_i$ is the Laplacian

$$\triangle_i = \frac{\partial^2}{\partial(x_i^1)^2},$$

$$\phi_{i,j}(|x_i - x_j|) = 2c\delta(x_i - x_j),$$

and $u^L(x)$ is the external field which keeps the system in the region L ($u^L(x) = 0$ if $x \in L$ and $u^L(x) = +\infty$ if $x \notin L$. Here $\phi_{i,j}(|x_i - x_j|)$ is symmetric.

Using semigroup theory we can reduce this problem to solution of system of equations:

$$i\frac{\partial \rho_s^L(x_1,\ldots,x_s;x_1',\ldots,x_s',t)}{\partial t} = [\mathsf{H}_s^L,\rho_s^L](x_1,\ldots,x_s;x_1',\ldots,x_s',t)$$

with initial date

$$\rho_s^L(x_1,\ldots,x_s;x_1',\ldots,x_s',t)|_{t=0} = \rho_s^L(x_1,\ldots,x_s;x_1',\ldots,x_s',0),$$

where $x_i \in \mathscr{R}$, $i = 1,\ldots,N$.

According to the theory of semigroups, the solution of the hierarchy of equations (4) has the form [8]:

$$(U^L(t)\rho^L)_s(x_1,..,x_s;x_1',..,x_s') =$$

$$(e^{\Omega(L)}e^{-i\mathsf{H}_s^L t}e^{-\Omega(L)}\rho^L e^{i\mathsf{H}_s^L t})_s(x_1,..,x_s;x_1',..,x_s')$$

which is equal for $x_1 < x_2 < \ldots < x_s, \quad x_1' < x_2' < \ldots < x_s'$ when $s = N$ to:

$$(e^{-i\mathsf{E}_s^L t}\rho^L e^{i\mathsf{E}_s^L t})_s(x_1,..,x_s;x_1',..,x_s'),$$

where

$$\mathsf{E}_s^L = \sum_{j=1}^{s=N} k_j^2,$$

$$(\Omega(L)\rho^L)_s(x_1,..,x_s;x_1',..,x_s') =$$

$$= \frac{N}{L}(1-\frac{s}{N})\int_L \sum_i \rho_{s+1}^L(x_1,..,x_s,x_{s+1};x_1',..,x_s',x_{s+1})\times$$

$$g_i^1(x_{s+1})\tilde{g}_i^1(x_{s+1})dx_{s+1},$$

$g_i^1(x_{s+1})$ is a complete orthonormal system of vectors in the one-particle space $\mathscr{L}_2(L)$,

$$\rho_s^L(x_1,...,x_s;x_1',...,x_s') = \sum_{\alpha=1} w_\alpha \psi_\alpha(x_1,...,x_s)\psi_\alpha^*(x_1',...,x_s')$$

is nuclear operator [7], and [2]:

$$\psi_\alpha(x_1,...x_s) = \psi_B(x_1,...x_s) = \sum_\sigma \mathsf{A}(\sigma)\sigma exp(i\sum_{j=1}^{s} k_j x_j).$$

To determine the solution

$$\rho_{\delta_s}^L(x_1,...,x_s;x_1',...,x_s',t) =$$

$$\sum_{\alpha=1} w_\alpha \psi_{\delta_\alpha}(x_1,...,x_s,t)\psi_{\delta_\alpha}^*(x_1',...,x_s',t)$$

of the Liouville's quantum kinetic equation with delta potential with the initial condition (2) we can use the method described in [3]. Namely, we use a Green's function

$$\psi_\delta(x_1,x_2,...,x_N,t) =$$

$$\sum_{\sigma\in S_N}\int_{\mathbb{R}}...\int_{\mathbb{R}} \mathsf{A}(\sigma)\prod_{j=1}^{N} e^{ik_{\sigma(j)}(x_j-y_{\sigma(j)})}e^{-it\sum_j \varepsilon(k_j)}dk_1...dk_N \quad (6)$$

where $\mathsf{A}(\sigma) = \prod\{S_{\alpha\beta} : \alpha\beta$ is an inversion in $\sigma\}$, $\sigma \in S_N$ be a permutation of $\{1,...,N\}$. Recall that an inversion in a permutation σ is an ordered pair $\{\sigma(i),\sigma(j)\}$ in which $i < j$ and $\sigma(i) > \sigma(j)$.

For case N=2 from equation (6) at $t = 0$ we obtain [3]:

$$e^{-i(k_2y_1+k_1y_2)} = -\frac{c-i(k_2-k_1)}{c+i(k_2-k_1)}e^{-i(k_1y_1+k_2y_2)} =$$

$$-exp(i\theta_{2,1})e^{-i(k_1y_1+k_2y_2)}. \quad (7)$$

3 Application

The result of Lieb-Liniger [2] and the equation (7) from [3] can be used to solve the problem of two keys. Namely, equations (6) and (7) make it possible to transmit, without the transfer of keys, a time-dependent and time-independent information, respectively.

Solution of the problem: Alice is trying to send a private message to Bob. Alice put plaintext for case $N = 3$ as $P = e^{-i(k_1y_1+k_2y_2+k_3y_3)}$ in a box and apply her lock $e_A = -e^{i\theta_{2,1}}$ on that box. She sends the locked box

$$-e^{i\theta_{2,1}}P = e^{-i(k_2y_1+k_1y_2+k_3y_3)}$$

to Bob, who then put his own lock $e_B = -e^{i\theta_{3,1}}$ to the box. Now Bob the box under the double lock sends back to Alice

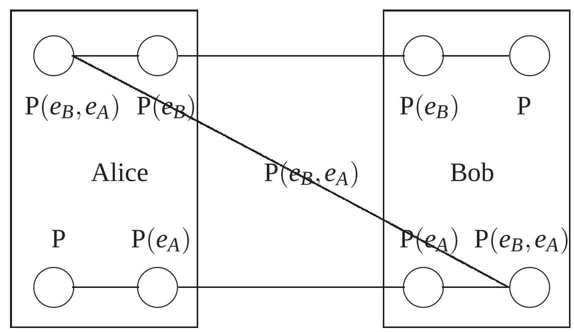
$$e^{i(\theta_{3,1}+\theta_{2,1})}P = e^{-i(k_2y_1+k_3y_2+k_1y_3)}.$$

Alice will open her lock $-e^{-i\theta_{2,1}} = -e^{i\theta_{1,2}} = -e^{-i\theta_{2,3}} = -e^{i\theta_{3,2}}$ before send it $-e^{i\theta_{3,2}}e^{-i(k_2y_1+k_3y_2+k_1y_3)} = e^{-i(k_3y_1+k_2y_2+k_1y_3)}$ back to Bob for the second time. And lastly Bob will unlock $-e^{-i\theta_{3,1}} = -e^{i\theta_{1,3}}$ his lock to open the plaintext

$$e^{i\theta_{1,3}}e^{-i(k_3y_1+k_2y_2+k_1y_3)} = e^{-i(k_1y_1+k_2y_2+k_3y_3)}$$

(see diagram below)

Here we used $-e^{i\theta_{2,1}} = -e^{i\theta_{2,3}}$ (Appendix), $\theta_{i,j} = -\theta_{j,i}$ [2] with factors: $-e^{-i\theta_{2,1}} = -e^{i\theta_{1,2}}$ which is equal to: $-e^{-i\theta_{2,3}} = -e^{i\theta_{3,2}}$.



4 Appendix

There are many different ways of transitions from set $(k_1k_2k_3)$ to set $(k_3k_2k_1)$ with equal probability [2]:

$$(k_1k_2k_3) \to (k_2k_1k_3) \to (k_2k_3k_1) \to (k_3k_2k_1)$$

and

$$(k_1k_2k_3) \to (k_1k_3k_2) \to (k_3k_1k_2) \to (k_3k_2k_1)$$

Corresponding to permutation $(k_1k_2k_3) \to (k_3k_2k_1)$ factors are $-e^{i\theta_{3,2}+i\theta_{3,1}+i\theta_{2,1}}$. Analogously factors $-e^{i\theta_{3,1}+i\theta_{2,1}}$ and $-e^{i\theta_{2,3}+i\theta_{3,1}}$ correspond to permutation $(k_1k_2k_3) \to (k_2k_3k_1)$. In other words:

$$e^{i\theta_{3,1}+i\theta_{2,1}}e^{-i(k_1y_1+k_2y_2+k_3y_3)} = e^{i\theta_{3,1}+i\theta_{2,3}}e^{-i(k_1y_1+k_2y_2+k_3y_3)}.$$

It follows that

$$-e^{i\theta_{2,1}} = -e^{i\theta_{2,3}}.$$

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