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Fractional Calculus of Wright Function with Raizada Polynomial

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Abstract: Here we aim at presenting fractional integral and derivative formulas of Saigo and Meada type, which are involved in a product of Wright function and Raizada polynomial. The results are obtained in a compact form containing the Riemann-Liouville, Erdelyi-Kober and Saigo operators of fractional calculus.

Keywords: Generalized fractional calculus, Raizada polynomial, Wright function.

1 Introduction

Each special function arises in one or more physical contexts as a solution of the differential equation that can be transformed into the hypergeometric function. The special function is then defined in terms of generalized hypergeometric function. The Wright function is one of special functions. The Wright function along with fractional calculus operators plays an important role in the theory of fractional calculus [1,2]. This approach is worth considering, and it is instructive to see that most of the special functions encountered in applied mathematics have a common root in their relation to the hypergeometric function. Here we establish theorems for the generalized fractional calculus operators, applied on the product of Wright function and Raizada polynomial. The solutions are constructed in generalized Wright function. We begin our study from the following definitions

Wright Function
E. M. Wright [3,4] has investigated the asymptotic behaviour of the sum

$$ {}_p\Psi_q(z) = \sum_{n=0}^{\infty} \frac{\Gamma(a_1+\beta_1 n)\ldots\Gamma(a_p+\beta_p n)}{\Gamma(\rho_1+\mu_1 n)\ldots\Gamma(\rho_q+\mu_q n)} \frac{z^n}{n!} $$

$$ = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+\beta_i n)}{\prod_{j=1}^{q}\Gamma(\rho_j+\mu_j n)} \frac{z^n}{n!} \tag{1} $$

for large $|z|$. Here the β_i and the μ_j are real, positive and

$$ 1+\sum_{j=1}^{q}\mu_j-\sum_{i=1}^{p}\beta_i > 0. \tag{2} $$

If all the μ_j and β_i are equal to unity, this reduces to a multiple of ${}_pF_q(z)$. (Further details are in Refs.[5,6,7,8,9,10,11, 12,13]).

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Raizada Polynomial

The generalized polynomial provides an elegant unified representation of the various known extension of the classical Hermite, Laguerre and Bessel polynomials [14,15,16,17] and is defined by the following Rodrigues type formula

$$S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l) = (At+B)^{-\alpha}(1-\tau t^r)^{\beta/\tau}T_{k,l}^{m+n}[(At+B)^{\alpha+qn}(1-\tau t^r)^{\frac{\beta}{\tau+sn}}] \tag{3}$$

with the differential operator $T_{k,l}$ being defined as

$$T_{k,l} \equiv t^l(k+tD_t), \tag{4}$$

where $D_t = \frac{d}{dt}$. The explicit form of this generalized polynomial set (check Ref.[16], p. 71, Eq. (2.34)) is

$$\begin{aligned} &S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l) \\ &= B^{qn}t^{l(m+n)}l^{m+n}(1-\tau t^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!} \\ &\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau t^r}{1-\tau t^r}\right)^p\left(\frac{At}{B}\right)^i. \end{aligned} \tag{5}$$

The manuscript is organized as follows. In Section 2 we give some details regarding the generalized fractional calculus operator. The fractional integration of the product ${}_p\Psi_q$-function and $S_n^{a,b,\tau}(\cdot)$ polynomial is mentioned in Section 3. Also, the fractional differential of the product ${}_p\Psi_q$ function and $S_n^{a,b,\tau}(\cdot)$ polynomial is given in Section 4. The content of the Section 5 is devoted to the fractional integro-differential of the product presented in the previous section. Finally, the conclusions are presented in Section 6.

2 Generalized Fractional Calculus Operator

The class of fractional operator equations of various types plays a very important role not only in mathematics but also in physics, control systems, dynamical systems, and engineering. Naturally, such equations are required to be solved. There are numerous studies focused in this direction [18,19,20,21,22,23,24,25,26,27,28,29]. An interesting and useful generalization of both the Riemann-Liouville and Erdelyi-Kober fractional integration operators has been introduced in terms of Gauss hypergeometric function [30] as given below.

Let $\alpha,\beta,\eta \in C$ and $x \in \Re_+$; then the generalized fractional integration and fractional differentiation operators associated with Gauss hypergeometric function [30,31,32] are defined as follows

$$(I_{0+}^{\alpha,\beta,\eta}f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt \tag{6}$$

$$(\Re(\alpha)>0;$$

$$= \frac{d^n}{dx^n}\left(I_{0+}^{\alpha+n,\beta-n,\eta-n}f\right)(x) \tag{7}$$

$$(\Re(\alpha)\le 0; n=[\Re(-\alpha)]+1);$$

$$\left(I_{-}^{\alpha,\beta,\eta}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)f(t)dt \tag{8}$$

$$(\Re(\alpha)>0;$$

$$= (-1)^n\frac{d^n}{dx^n}\left(I_{-}^{\alpha+n,\beta-n,\eta}f\right)(x)$$

$$(\Re(\alpha)\le 0; n=[\Re(-\alpha)]+1);$$

and

$$\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x) = \left(I_{0+}^{-\alpha,-\beta,\alpha+\eta} f\right)(x) = \frac{d^n}{dx^n}\left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f\right)(x) \tag{9}$$

$$(\Re(\alpha) > 0; n = [\Re(\alpha)] + 1);$$

$$\left(D_{-}^{\alpha,\beta,\eta} f\right)(x) = \left(I_{-}^{-\alpha,-\beta,\alpha+\eta} f\right)(x) = (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} f\right)(x) \tag{10}$$

$$(\Re(\alpha) > 0; n = [\Re(\alpha)] + 1).$$

The Riemann-Liouville, Weyl and Erdelyi-Kober fractional calculus operators are recovered as special cases of the operators $I_{+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$ as shown below

$$(R_{0,x}^{\alpha} f)(x) = \left(I_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt \tag{11}$$

$$(\Re(\alpha) > 0);$$

$$= \frac{d^n}{dx^n}\left(R_{0,x}^{\alpha+n} f\right)(x) \tag{12}$$

$$(0 < \Re(\alpha) + n \leq 1; n = 1, 2, 3, \ldots);$$

$$(W_{x,\infty}^{\alpha}) = (I_{-}^{\alpha,-\alpha,\eta} f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1} f(t)dt \tag{13}$$

$$(\Re(\alpha) > 0;$$

$$= (-1)^n \frac{d^n}{dx^n}(W_{x,\infty}^{\alpha+n} f)(x) \tag{14}$$

$$(0 < \Re(\alpha) + n \leq 1; n = 1, 2, \ldots);$$

$$(E_{0,x}^{\alpha,\eta} f)(x) = (I_{0+}^{\alpha,0,\eta} f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)dt \tag{15}$$

$$(\Re(\alpha) > 0);$$

$$(K_{x,\infty}^{\alpha,\eta} f)(x) = (I_{-}^{\alpha,0,\eta} f)(x) = \frac{x^{\eta}}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)dt. \tag{16}$$

$$(\Re(\alpha) > 0).$$

Now the definition of the following generalized fractional integration and differentiation operators of any complex order involve Appell function F_3 (see Ref. [33], p.393, Eqs. (4.12) ad (4.13)) in the kernel in the following form.

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x t^{-\alpha'}(x-t)^{\gamma-1} F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}) f(t)dt \tag{17}$$

$$(\Re(\gamma) > 0);$$

$$= \frac{d^n}{dx^n}(I_{0+}^{\alpha,\alpha',\beta+n,\beta',\gamma+n} f)(x). \tag{18}$$

$$(\Re(\gamma) \leq 0; n = [-\Re(\gamma)] + 1);$$

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_{x}^{\infty}t^{-\alpha}(t-x)^{\gamma-1}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x})f(t)dt \tag{19}$$

$$(\Re(\gamma)>0);$$

$$=(-1)^n\frac{d^n}{dx^n}(I_{-}^{\alpha,\alpha',\beta,\beta'+n,\gamma+n}f)(x) \tag{20}$$

$$(\Re(\gamma)\le 0;\ n=[-\Re(\gamma)]+1);$$

and

$$(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f)(x)=(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f)(x) \tag{21}$$

$$=\frac{d^n}{dx^n}\left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n}f\right)(x) \tag{22}$$

$$(\Re(\gamma)>0;\ n=[\Re(\gamma)]+1);$$

$$(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f)(x)=(I_{-}^{-\alpha',\alpha,-\beta',-\beta,-\gamma}f)(x) \tag{23}$$

$$=(-1)^n\frac{d^n}{dx^n}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n}f\right)(x) \tag{24}$$

$$(\Re(\gamma)>0;\ n=[\Re(\gamma)]+1).$$

These operators are reduced to that in (17) - (23) as the following.

$$(I_{0+}^{\alpha,0,\beta,\beta',\gamma}f)(x)=(I_{0+}^{\gamma,\alpha-\gamma,-\beta}f)(x)(\gamma\in C); \tag{25}$$

$$(I_{-}^{\alpha,0,\beta,\beta',\gamma}f)(x)=(I_{-}^{\gamma,\alpha-\gamma,-\beta}f)(x)(\gamma\in C); \tag{26}$$

$$(D_{0+}^{0,\alpha',\beta,\beta',\gamma}f)(x)=(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f)(x)(R(\gamma)>0); \tag{27}$$

$$(D_{-}^{0,\alpha',\beta,\beta',\gamma}f)(x)=(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f)(x)(R(\gamma)>0). \tag{28}$$

(Further in Refs. ([33], p. 394, eqs (4.18) and (4.19)), we also have

$$(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}f)(x)=$$

$$\Gamma\begin{bmatrix}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha'\\ \rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta,\rho+\beta'\end{bmatrix}x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{29}$$

where $\Re(\gamma)>0,\ \Re(\rho)>\max[0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')]$
and

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}f)(x)=$$

$$\Gamma\begin{bmatrix}1+\alpha+\alpha'-\gamma-\rho,1+\alpha+\beta'-\gamma-\rho,1-\beta-\rho\\ 1-\rho,1+\alpha+\alpha'+\beta'-\gamma-\rho,1+\alpha-\beta-\rho\end{bmatrix}x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{30}$$

where $R(\gamma)>0,\ \Re(\rho)<1+\min[\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha-\beta'-\gamma)]$.

Here the symbol $\Gamma\begin{bmatrix}a,b,c\\ d,e,f\end{bmatrix}$ is employed to represent the ratios of product of gamma function $\dfrac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}$.

3 Fractional Integration of the Product ${}_p\Psi_q$-Function and $S_n^{a,b,\tau}(\cdot)$ Polynomial

Theorem 1. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho \in C, Re(\gamma)>0$ and $Re(\rho)>0$ and let $a\in C,\ \sigma>0$. If the condition of equation (2) and (29) are satisfied, then the fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and the following relation holds.*

$$\begin{aligned}
&(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\
&=x^{\rho+l(m+n)-\alpha-\alpha'+\gamma-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\\
&\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i
\end{aligned}$$

$$\times{}_{p+3}\Psi_{q+3}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p},(\theta,\sigma),(\theta+\gamma-\alpha-\alpha'-\beta,\sigma),(\theta+\beta'-\alpha',\sigma)\\(a_j,b_j)_{1,q},(\theta+\gamma-\alpha-\alpha',\sigma),(\theta+\gamma-\alpha'-\beta,\sigma)(\theta+\beta',\sigma)\end{matrix}\right.\right] \tag{31}$$

Where $\theta=\rho+l(m+n)+rw+rp+i$.

*Proof.*With equation (1), (5) and (17)

$$(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)$$

$$=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x t^{-\alpha'}(x-t)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)$$

$$\times[t^{\rho-1}{}_p\Psi_q(at^{\sigma})S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)]dt$$

$$=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x t^{-\alpha'}(x-t)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)$$

$$\times\left\{t^{\rho-1}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{(at^{\sigma})^k}{k!}B^{qn}t^{l(m+n)}l^{m+n}(1-\tau t^r)^{sn}\right.$$

$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$

$$\left.\times\left(\frac{j+k+re}{l}\right)_{m+n}\frac{(-\tau t^r)^p}{(1-\tau t^r)^p}\left(\frac{At}{B}\right)^i\right\}dt$$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}B^{qn}l^{m+n}\sum_{w=0}^{\infty}\frac{(\tau)^w}{w!}(-sn+p)_w$$

$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}$$

$$\times\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{i+k+re}{l}\right)_{m+n}(-\tau)^p\left(\frac{A}{B}\right)^i\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x t^{-\alpha'}(x-t)^{\gamma-1}$$

$$\times F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)t^{\rho+\sigma k+l(m+n)+rw+rp+i-1}dt$$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}B^{qn}l^{m+n}\sum_{w=0}^{\infty}\frac{(\tau)^w}{w!}(-sn+p)_w$$

$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}$$

$$\times\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{i+k+re}{l}\right)_{m+n}(-\tau)^p\left(\frac{A}{B}\right)^i I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho+\sigma k+l(m+n)+rw+rp+i-1}](x).$$

Using (29) in the above expression we obtain

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}B^{qn}l^{m+n}\sum_{w=0}^{\infty}\frac{(\tau)^w}{w!}(-sn+p)_w$$

$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}$$

$$\times\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{i+k+re}{l}\right)_{m+n}(-\tau)^p\left(\frac{A}{B}\right)^i$$

$$\times\frac{\Gamma(\theta+\sigma k)\Gamma(\theta+\sigma k+\gamma-\alpha-\alpha'-\beta)\Gamma(\theta+\sigma k+\beta'-\alpha')}{\Gamma(\theta+\sigma k+\gamma-\alpha-\alpha')\Gamma(\theta+\sigma k+\gamma-\alpha'-\beta)\Gamma(\theta+\sigma k+\beta')}$$

$$\times x^{\rho+\sigma k+l(m+n)+rw+rp+i-\alpha-\alpha'+\gamma-1}$$

$$=x^{\rho+l(m+n)-\alpha-\alpha'+\gamma-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}$$

$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$

$$\times\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times{}_{p+3}\Psi_{q+3}\left[ax^\sigma\left|\begin{matrix}(a_i,b_i)_{1,p},(\theta,\sigma),(\theta+\gamma-\alpha-\alpha'-\beta,\sigma),(\theta+\beta'-\alpha',\sigma)\\(a_j,b_j)_{1,q},(\theta+\gamma-\alpha-\alpha',\sigma),(\theta+\gamma-\alpha'-\beta,\sigma)(\theta+\beta',\sigma)\end{matrix}\right.\right]. \tag{32}$$

which is a required result.
If $\alpha'=0$ in (31), then with (25), we arrive at

Corollary 1. *Let $\alpha,\beta,\eta,\rho\in C$, $Re(\alpha)>0$ and $Re(\rho)>0$ and let $a\in C$, $\sigma>0$. If the condition (2) is satisfied, then the fractional integral $I_{0+}^{\alpha,\beta,\gamma}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and the following relation holds.*

$$(I_{0+}^{\alpha,\beta,\eta}[t^{\rho-1}{}_p\Psi_q(at^\sigma)s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)$$

$$=x^{\rho-\beta+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times {}_{p+2}\Psi_{q+2}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p}, & (\theta,\sigma), & (\theta+\eta-\beta,\sigma)\\ (a_j,b_j)_{1,q}, & (\theta-\beta,\sigma), & (\theta+\alpha+\eta,\sigma)\end{matrix}\right.\right]. \tag{33}$$

If we go ahead with the equations (11) and (33), then we get

$$(I_{0+}^{\alpha}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)$$

$$= x^{\rho+\alpha+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times {}_{p+1}\Psi_{q+1}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p}, & (\theta,\sigma)\\ (a_j,b_j)_{1,q}, & (\theta+\alpha,\sigma)\end{matrix}\right.\right]. \tag{34}$$

If we re-do equation (33) with (15), we obtain

$$(E_{0+}^{\alpha,\eta}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)$$

$$= x^{\rho+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times {}_{p+1}\Psi_{q+1}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p}, & (\theta+\eta,\sigma)\\ (a_j,b_j)_{1,q}, & (\theta+\alpha+\eta,\sigma)\end{matrix}\right.\right]. \tag{35}$$

Equations (34) and (35) move along the path shown in the equation (2).

Theorem 2. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho\in C, Re(\gamma)>0$ and $Re(\rho)>0$ and let $a\in C,\ \sigma>0$. If the condition of (2) and (30)are satisfied, then the fractional integral $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and the following relation holds.*

$$\begin{aligned}&I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\&= x^{\rho+l(m+n)-\alpha-\alpha'+\gamma-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\\&\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\&\times\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\end{aligned}$$

$$\times {}_{p+3}\Psi_{q+3}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p},(1+\alpha+\alpha'-\gamma-\theta,-\sigma),(1+\alpha+\beta'-\gamma-\theta,-\sigma),(1-\beta-\theta,-\sigma)\\ (a_j,b_j)_{1,q},(1-\theta,-\sigma),(1+\alpha+\alpha'+\beta'-\gamma-\theta,-\sigma),(1+\alpha-\beta-\theta,-\sigma)\end{matrix}\right.\right]. \tag{36}$$

*Proof.*Using equation (1), (5) and (19), we have

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)$$
$$=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^{\infty}t^{-\alpha}(t-x)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)$$
$$\times[t^{\rho-1}{}_p\Psi_q(at^{\sigma})S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)]dt$$
$$=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^{\infty}t^{-\alpha}(t-x)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)$$
$$\times\left\{t^{\rho-1}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{(at^{\sigma})^k}{k!}B^{qn}t^{l(m+n)}l^{m+n}(1-\tau t^r)^{sn}\right.$$
$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$
$$\left.\times\left(\frac{j+k+re}{l}\right)_{m+n}\frac{(-\tau t^r)^p}{(1-\tau t^r)^p}\left(\frac{At}{B}\right)^i\right\}dt$$
$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}B^{qn}l^{m+n}\sum_{w=0}^{\infty}\frac{(\tau)^w}{w!}(-sn+p)_w$$
$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}$$
$$\times\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{i+k+re}{l}\right)_{m+n}(-\tau)^p\left(\frac{A}{B}\right)^iI_{-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho+\sigma k+l(m+n)+rw+rp+i-1}](x).$$

Applying the formula (30), the above expression becomes

$$=x^{\rho+l(m+n)-\alpha-\alpha'+\gamma-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}$$
$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$
$$\times\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$
$$\times{}_{p+3}\Psi_{q+3}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p},(1+\alpha+\alpha'-\gamma-\theta,-\sigma),(1+\alpha+\beta'-\gamma-\theta,-\sigma),(1-\beta-\theta,-\sigma)\\(a_j,b_j)_{1,q},(1-\theta,-\sigma),(1+\alpha+\alpha'+\beta'-\gamma-\theta,-\sigma),(1+\alpha-\beta-\theta,-\sigma)\end{matrix}\right.\triangleright\right].\qquad(37)$$

which is a required result.

If we take $\alpha'=0$ in (36), then with (26), we arrive at

Corollary 2. *Let $\alpha,\beta,\eta,\rho\in C$, $Re(\alpha)>0$ and $Re(\rho)>0$ and let $a\in C$, $\sigma>0$. If the condition (2) is satisfied, then the fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and the following relation holds.*

$$(I_{-}^{\alpha,\beta,\eta}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)$$
$$=x^{\rho-\beta+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}$$
$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times {}_{p+2}\Psi_{q+2}\left[ax^{\sigma}\middle|\begin{matrix}(a_i,b_i)_{1,p}, & (1+\beta-\theta,-\sigma), & (1+\eta-\theta,-\sigma)\\ (a_j,b_j)_{1,q}, & (1-\theta,-\sigma), & (1+\alpha+\beta+\eta-\theta,-\sigma)\end{matrix}\right]. \tag{38}$$

Now, if we set $\beta=-\alpha$, then we can show equation (38)as we did in (13)

$$\begin{aligned}&(W^{\alpha}_{x,\infty}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\ &=x^{\rho+\alpha+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}\\ &\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\\ &\times {}_{p+1}\Psi_{q+1}\left[ax^{\sigma}\middle|\begin{matrix}(a_i,b_i)_{1,p}, & (1-\alpha-\theta,-\sigma)\\ (a_j,b_j)_{1,q}, & (1-\theta,-\sigma)\end{matrix}\right].\end{aligned} \tag{39}$$

Next, If we take $\beta=0$, (38) implies that

$$\begin{aligned}&(K^{\alpha,\eta}_{x,\infty}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\ &=x^{\rho+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}\\ &\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\\ &\times {}_{p+1}\Psi_{q+1}\left[ax^{\sigma}\middle|\begin{matrix}(a_i,b_i)_{1,p}, & (1+\eta-\theta,-\sigma)\\ (a_j,b_j)_{1,q}, & (1+\alpha+\eta-\theta,-\sigma)\end{matrix}\right].\end{aligned} \tag{40}$$

Equations (39) and (40) move along the path shown in the equation (2).

4 Fractional Differential of the Product ${}_p\Psi_q$ Function and $S_n^{a,b,\tau}(\cdot)$ Polynomial

Theorem 3. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho\in C, Re(\gamma)>0$ and $Re(\rho)>0$ and let $a\in C,\ \sigma>0$. If the condition of (2) and (29) are satisfied, then the fractional integral $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and the following relation holds.*

$$\begin{aligned}&(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\ &=x^{\rho-\gamma+\alpha+\alpha'+l(m+n)-1}B^{qn}l^{m+n}(1-\tau t^r)^{sn}\\ &\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\ &\times\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\end{aligned}$$

$$\times {}_{p+3}\Psi_{q+3}\left[ax^{\sigma}\middle|\begin{matrix}(a_i,b_i)_{1,p},(\theta,\sigma)(\theta-\gamma+\alpha+\alpha'+\beta',\sigma),(\theta-\beta+\alpha,\sigma)\\ (a_j,b_j)_{1,q},(\theta-\gamma+\alpha+\alpha',\sigma),(\theta-\gamma+\alpha+\beta',\sigma),(\theta-\beta,\sigma)\end{matrix}\right]. \tag{41}$$

Proof. With (1), (5) and (22), we have

$$\begin{aligned}&(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\ &=\frac{d^k}{dx^k}(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})S_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\end{aligned}$$

where $k=[R(\gamma)+1]$.

$$
\begin{aligned}
&=\frac{d^k}{dx^k}(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}(t^{\rho-1}\\
&\times\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{(at^{\sigma})^k}{k!}B^{qn}t^{l(m+n)}l^{m+n}(1-\tau t^r)^{sn}\\
&\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau t^r}\right)^p\left(\frac{Ax}{B}\right)^i)(x))\\
&=B^{qn}l^{m+n}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\\
&\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\\
&\times\frac{(-sn+p)_w}{w!}\frac{d^k}{dx^k}\left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}[t^{\rho+\sigma k+l(m+n)+rw+rp+i-1}]\right)(x)\\
&=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{A}{B}\right)^i\\
&\times\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\frac{(-sn+p)_w}{w!}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}\\
&\times\frac{\Gamma(\theta+\sigma k)\Gamma(\theta+\sigma k-\gamma+\alpha+\alpha'+\beta')\Gamma(\theta+\sigma k-\beta+\alpha)}{\Gamma(\theta+\sigma k+\alpha+\alpha'-\gamma+k)\Gamma(\theta+\sigma k-\gamma+\alpha+\beta')\Gamma(\theta+\sigma k-\beta)}\\
&\frac{d^k}{dx^k}x^{\theta+\alpha+\alpha'-\gamma+k+\sigma k-1}.
\end{aligned}
$$

Using $\frac{d^n}{dx^n}x^m=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$, where $m\geq n$ in the above expression, we obtain

$$
\begin{aligned}
&=x^{\rho-\gamma+\alpha+\alpha'+l(m+n)-1}B^{qn}l^{m+n}(1-\tau t^r)^{sn}\\
&\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i
\end{aligned}
$$

$$
\times_{p+3}\Psi_{q+3}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p},(\theta,\sigma)(\theta-\gamma+\alpha+\alpha'+\beta',\sigma),(\theta-\beta+\alpha,\sigma)\\(a_j,b_j)_{1,q},(\theta-\gamma+\alpha+\alpha',\sigma),(\theta-\gamma+\alpha+\beta',\sigma),(\theta-\beta,\sigma)\end{matrix}\right.\right]. \tag{42}
$$

With the help of (41) and (27), we arrive at

Corollary 3. *Let $\alpha,\beta,\eta,\rho \in C$, $Re(\alpha) > 0$ and $Re(\rho) > 0$ and let $a \in C$, $\sigma > 0$. If the condition (2)is satisfied, then the fractional integral $D_{0+}^{\alpha,\beta,\eta}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and following relation holds.*

$$(D_{+}^{\alpha,\beta,\eta}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)$$
$$= x^{\rho+\beta+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}$$
$$\times \frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times {}_{p+2}\Psi_{q+2}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p}, & (\theta,\sigma), & (\theta+\alpha+\beta+\eta,\sigma)\\ (a_j,b_j)_{1,q}, & (\theta+\beta,\sigma), & (\theta+\eta,\sigma)\end{matrix}\right.\right]. \tag{43}$$

Theorem 4. *Let $\alpha,\alpha',\beta,\beta;\gamma,\rho \in C$, $Re(\gamma) > 0$ and $Re(\rho) > 0$ and let $a \in C$, $\sigma > 0$. If the condition of (2) and (30) are satisfied, then the fractional integral $D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and the following relation holds.*

$$(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;x,s,q,A,B,m,k,l)])(x)$$
$$= x^{\rho-\gamma+\alpha+\alpha'+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$
$$\times \frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times {}_{p+3}\Psi_{q+3}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p},(1-\alpha'-\alpha+\gamma-\theta,-\sigma),(1-\alpha'-\beta+\gamma-\theta,-\sigma),(1+\beta'-\theta,-\sigma)\\ (a_j,b_j)_{1,q},(1-\theta,-\sigma),(1-\alpha'-\alpha-\beta+\gamma-\theta,-\sigma),(1-\alpha'+\beta'-\theta,-\sigma)\end{matrix}\right.\right]. \tag{44}$$

Proof. Using (1), (5) and (24), we arrive at

$$(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;x,s,q,A,B,m,k,l)])(x)$$
$$= (-1)^k\frac{d^k}{dx^k}(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;x,s,q,A,B,m,k,l)])(x)$$

where $k=[R(\gamma)+1]$.

$$=(-1)^k\frac{d^k}{dx^k}\left[I_-^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}\left(t^{\rho-1}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{(at^{\sigma})^k}{k!}B^{qn}t^{l(m+n)}l^{m+n}\right.\right.$$

$$\times(1-\tau t^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$

$$\left.\left.\times\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau t^r}{1-\tau t^r}\right)^p\left(\frac{At}{B}\right)^i\right)(x)\right]$$

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$

$$\times(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}(-1)^k\frac{d^k}{dx^k}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}$$

$$\times\left(I_-^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}\left[t^{\rho+\sigma k+rw+rp+l(m+n)+i-1}\right]\right)(x).$$

Applying the formula (30)

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$

$$\times\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}a^k$$

$$\times\frac{\Gamma(1-\alpha'-\alpha+\gamma-k-\theta-\sigma k)\Gamma(1-\alpha'-\beta+\gamma-\theta-\sigma k)\Gamma((1+\beta'-\theta-\sigma k)}{\Gamma(1-\theta-\sigma k)\Gamma(1-\alpha'-\alpha-\beta+\gamma-\theta-\sigma k)\Gamma(1-\alpha'+\beta'-\theta-\sigma k)}$$

$$\times(-1)^k\frac{d^k}{dx^k}x^{\alpha+\alpha'-\gamma+k+\theta+\sigma k-1}$$

$$=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$

$$\times\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}$$

$$\times\frac{\Gamma(1-\alpha'-\alpha+\gamma-k-\theta-\sigma k)\Gamma(1-\alpha'-\beta+\gamma-\theta-\sigma k)\Gamma((1+\beta'-\theta-\sigma k)}{\Gamma(1-\theta-\sigma k)\Gamma(1-\alpha'-\alpha-\beta+\gamma-\theta-\sigma k)\Gamma(1-\alpha'+\beta'-\theta-\sigma k)}$$

$$\times(1-\alpha'-\alpha+\gamma-k-\theta-\sigma k)_k\ x^{\alpha+\alpha'-\gamma+\theta+\sigma k}$$

$$=x^{\rho-\gamma+\alpha+\alpha'+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times_{p+3}\Psi_{q+3}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p},(1-\alpha'-\alpha+\gamma-\theta,-\sigma),(1-\alpha'-\beta+\gamma-\theta,-\sigma),(1+\beta'-\theta,-\sigma)\\(a_j,b_j)_{1,q},(1-\theta,-\sigma),(1-\alpha'-\alpha-\beta+\gamma-\theta,-\sigma),(1-\alpha'+\beta'-\theta,-\sigma)\end{matrix}\right.\right]. \quad (45)$$

By using relation (28) in (44), we arrive at

Corollary 4. *Let $\alpha,\beta,\eta,\rho \in C$, $Re(\alpha)>0$ and $Re(\rho)>0$ and let $a\in C$, $\sigma>0$. If the condition (2) is satisfied, then the fractional integral $D_{-}^{\alpha,\beta,\eta}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and the following relation holds.*

$$\begin{aligned}&(D_{-}^{\alpha,\beta,\eta}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\&=x^{\rho+\beta+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}\\&\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\end{aligned}$$

$$\times_{p+2}\Psi_{q+2}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p}, & (1-\beta-\theta,-\sigma), & (1+\alpha+\eta-\theta,-\sigma)\\(a_j,b_j)_{1,q}, & (1-\theta,-\sigma), & (1-\beta+\eta-\theta,-\sigma)\end{matrix}\right.\right]. \quad (46)$$

5 Fractional Integro-Differential of the Product ${}_p\Psi_q$-Function and $S_n^{a,b,\tau}(\cdot)$ Polynomial

Theorem 5. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho\in C, Re(\gamma)>0$ and $Re(\rho)>0$ and let $a\in C,\ \sigma>0$. If the condition of (2) and (29) are satisfied, then the fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and the following relation holds.*

$$\begin{aligned}&(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\&=x^{\rho+l(m+n)-\alpha-\alpha'+\gamma-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\\&\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\&\times\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\end{aligned}$$

$$\times_{p+3}\Psi_{q+3}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p},(\theta,\sigma),(\theta+\gamma-\alpha-\alpha'-\beta,\sigma),(\theta+\beta'-\alpha',\sigma)\\(a_j,b_j)_{1,q},(\theta+\gamma-\alpha-\alpha',\sigma),(\theta+\gamma-\alpha'-\beta,\sigma)(\theta+\beta',\sigma)\end{matrix}\right.\right]. \quad (47)$$

Proof. With (1), (5) and (18)

$$
\begin{aligned}
&(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right])(x)\\
&=\frac{d^k}{dx^k}(I_{0+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k}\left[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right](x)\\
&=\frac{d^k}{dx^k}(I_{0+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k}t^{\rho-1}(\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{(at^{\sigma})^k}{k!}B^{qn}t^{l(m+n)}l^{m+n}(1-\tau t^r)^{sn}\\
&\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau t^r}{1-\tau t^r}\right)^p\left(\frac{At}{B}\right)^i)(x)\\
&=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}\frac{d^k}{dx^k}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}\\
&\times(I_{0+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k}(t^{\rho+\sigma k+rw+rp+l(m+n)+i-1})(x)
\end{aligned}
$$

Using the formula (29)

$$
\begin{aligned}
&=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\frac{(sn+p)_w}{w!}\left(\frac{A}{B}\right)^i\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}\\
&\times\frac{\Gamma(\theta+\sigma k)\Gamma(\theta+\sigma k+\gamma-\alpha-\alpha'-\beta)\Gamma(\theta+\sigma k+\beta'-\alpha')}{\Gamma(\theta+\sigma k+\gamma+k-\alpha-\alpha')\Gamma(\theta+\sigma k+\gamma-\alpha'-\beta)\Gamma(\theta+\sigma k+\beta')}\\
&\times\frac{d^k}{dx^k}x^{\gamma+k-\alpha-\alpha'+\theta+\sigma k-1}
\end{aligned}
$$

Finally using $\frac{d^n}{dx^n}x^m=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$, where $m\geq n$, the above expression becomes

$$
\begin{aligned}
&x^{\rho+\gamma-\alpha-\alpha'+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\\
&\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i(x)
\end{aligned}
$$

$$
\times{}_{p+3}\Psi_{q+3}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p},(\theta,\sigma),(\theta+\gamma-\alpha-\alpha'-\beta,\sigma),(\theta+\beta'-\alpha',\sigma)\\(a_j,b_j)_{1,q},(\theta+\gamma-\alpha-\alpha',\sigma),(\theta+\gamma-\alpha'-\beta,\sigma),(\theta+\beta',\sigma)\end{matrix}\right.\right] \qquad (48)
$$

If we take $\alpha' = 0$ in (47), we arrive at

Corollary 5. *Let $\alpha, \beta, \eta, \rho \in C$, $Re(\alpha) > 0$ and $Re(\rho) > 0$ and let $a \in C$, $\sigma > 0$. If the condition (2) is satisfied, then the fractional integral $I_{0+}^{\alpha,\beta,\gamma}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and following relation holds.*

$$\begin{aligned}
&(I_{0+}^{\alpha,\beta,\eta}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\
&= x^{\rho-\beta+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}\\
&\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\\
&\times{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_i,b_i)_{1,p}, & (\theta,\sigma), & (\theta+\eta-\beta,\sigma)\\ (a_j,b_j)_{1,q}, & (\theta,-\beta,\sigma), & (\theta+\alpha+\eta,\sigma)\end{matrix}\right].
\end{aligned} \tag{49}$$

Theorem 6. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in C, Re(\gamma) > 0$ and $Re(\rho) > 0$ and let $a \in C$, $\sigma > 0$. If the condition of (2) and (30) are satisfied, then the fractional integral $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$exists and following relation holds.*

$$\begin{aligned}
&I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)\\
&= x^{\rho+l(m+n)-\alpha-\alpha'+\gamma-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\\
&\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i\\
&\times{}_{p+3}\Psi_{q+3}\left[ax^{\sigma}\left|\begin{matrix}(a_i,b_i)_{1,p},(1+\alpha+\alpha'-\gamma-\theta,-\sigma),(1+\alpha+\beta'-\gamma-\theta,-\sigma),(1-\beta-\theta,-\sigma)\\ (a_j,b_j)_{1,q},(1-\theta,-\sigma),(1+\alpha+\alpha'+\beta'-\gamma-\theta,-\sigma),(1+\alpha-\beta-\theta,-\sigma)\end{matrix}\right.\right].
\end{aligned} \tag{50}$$

Proof. By using (1), (5) and (20), we get

$$\begin{aligned}
&(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k}\left[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right])(x)\\
&=\frac{d^k}{dx^k}(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k}\left[t^{\rho-1}{}_p\Psi_q(at^{\sigma})s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)\right](x)\\
&=\frac{d^k}{dx^k}(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k}t^{\rho-1}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{(at^{\sigma})^k}{k!}B^{qn}t^{l(m+n)}l^{m+n}(1-\tau t^r)^{sn}\\
&\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau t^r}{1-\tau t^r}\right)^p\left(\frac{At}{B}\right)^i(x)\\
&=B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\\
&\times\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\left(\frac{A}{B}\right)^i\frac{(-sn+p)_w}{w!}\frac{d^k}{dx^k}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}\\
&\times(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k}t^{\rho+\sigma k+rw+rp+l(m+n)+i-1})(x)
\end{aligned}$$

Using the formula (30)

$$= B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$

$$\times\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\frac{(sn+p)_w}{w!}\left(\frac{A}{B}\right)^i\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}$$

$$\times\frac{\Gamma(1+\alpha+\alpha'-\gamma-k-\theta-\sigma k)\Gamma(1+\alpha+\beta'-\gamma-\theta-\sigma k)\Gamma(1-\beta-\theta-\sigma k)}{\Gamma(1-\theta-\sigma k)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\theta-\sigma k)\Gamma(1+\alpha-\beta-\theta-\sigma k)}$$

$$\times(-1)^k\frac{d^k}{dx^k}x^{\gamma+k-\alpha-\alpha'+\theta+\sigma k-1}$$

$$= B^{qn}l^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\sum_{w=0}^{\infty}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$

$$\times\left(\frac{j+k+re}{l}\right)_{m+n}(-\tau)^p\tau^w\frac{(sn+p)_w}{w!}\left(\frac{A}{B}\right)^i\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{a^k}{k!}$$

$$\times\frac{\Gamma(1+\alpha+\alpha'-\gamma-k-\theta-\sigma k)\Gamma(1+\alpha+\beta'-\gamma-\theta-\sigma k)\Gamma(1-\beta-\theta-\sigma k)}{\Gamma(1-\theta-\sigma k)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\theta-\sigma k)\Gamma(1+\alpha-\beta-\theta-\sigma k)}$$

$$\times(1+\alpha+\alpha'-\gamma-k-\theta-\sigma k)_k x^{\gamma-\alpha-\alpha'-\theta+\sigma k}$$

$$= x^{\rho+\gamma-\alpha-\alpha'+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}$$

$$\times\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-i)_j(a)_i}{p!i!j!e!}\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p$$

$$\times\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i(x)$$

$$\times{}_{p+3}\Psi_{q+3}\left[ax^\sigma\left|\begin{matrix}(a_i,b_i)_{1,p},(1+\alpha+\alpha'-\gamma-\theta,-\sigma),(1+\alpha+\beta'-\gamma-\theta,-\sigma),(1-\beta-\theta,-\sigma)\\(a_j,b_j)_{1,q},(1-\theta,-\sigma),(1+\alpha+\alpha'-\beta'-\gamma-\theta,-\sigma),(1+\alpha,-\beta-\theta,-\sigma)\end{matrix}\right.\right]. \tag{51}$$

which is a required result.

If we take $\alpha'=0$ in (50), we arrive at

Corollary 6.*Let $\alpha,\beta,\eta,\rho\in C$, $Re(\alpha)>0$ and $Re(\rho)>0$ and let $a\in C$, $\sigma>0$. If the condition (2) is satisfied, then the fractional integral of $I_{-}^{\alpha,\beta,\eta}$ of the product ${}_p\Psi_q$ and $S_n^{a,b,\tau}(\cdot)$ exists and the following relation holds.*

$$(I_{-}^{\alpha,\beta,\eta}[t^{\rho-1}{}_p\Psi_q(at^\sigma)s_n^{a,b,\tau}(t;r,s,q,A,B,m,k,l)])(x)$$

$$= x^{\rho-\beta+l(m+n)-1}B^{qn}l^{m+n}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{i=0}^{m+n}\sum_{j=0}^{i}\frac{(-1)^i(-p)_e(-1)_j(a)_i}{p!i!j!e!}$$

$$\times\frac{(-a-qn)_j}{(1-a-i)_j}\left(\frac{-b}{\tau}-sn\right)_p\left(\frac{j+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^i$$

$$\times{}_{p+2}\Psi_{q+2}\left[ax^\sigma\left|\begin{matrix}(a_i,b_i)_{1,p}, & (1+\beta-\theta,-\sigma), & (1+\eta-\theta,-\sigma)\\(a_j,b_j)_{1,q}, & (1-\theta,-\sigma), & (1+\alpha+\beta+\eta-\theta,-\sigma)\end{matrix}\right.\right]. \tag{52}$$

6 Conclusion

In the present paper, we have given the six theorems of generalized fractional integral and derivative operators given by Saigo and Meada. The theorems have been developed in terms of the product of Wright function and Raizada polynomial in a compact and elegant form with the help of Saigo-Meada power function formulas. The Wright function, expressed in this paper, is relatively basic in nature. Therefore on some suitable adjustment of the parameters on function, we may obtain other special functions such as M-series, Mittag-Leffler function, Bessel-Maitland function as its special cases. The results presented in this paper are easily converted in terms of a similar type of new integrals with different arguments after some suitable parametric replacements. In this sequel, one can obtain integral representation of more generalized special functions, which has a wide application in physics and engineering sciences.

References

[1] A. A. Kilbas, M. Saigo and J. J. Trujillo, On the generalized Wright function, *Fract. Calc. Appl. Anal.* **4**, 437–460 (2002).
[2] A. A. Kilbas, Fractional calculus of generalized Wright function, *Fract. Calc. Appl. Anal.* **8(2)**, 113–126 (2005).
[3] E. M. Wright, The asymptotic expension of the generalized hypergeometric function, *J. London Math. Soc.* **10**, 287–293 (1935).
[4] E. M. Wright, The generalized Bessel function of order greater than one, *Quart. J. Math.* **11(1)**, 36–48 (1940).
[5] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, McGraw Hill, New York, NY, USA, 1953.
[6] R. Gorenflo, Y. Luchko and F. Mainardi, Analytic properties and applications of Wright function, *Fract. Calc. Appl. Anal.* **4(2)**, 383–414 (1999).
[7] Yu. F. Lucko, Asymptotics of zeros of the Wright function, *Z. Anal. Anwendun.* **19**, 583–595 (2000).
[8] G. Pagnini and E. Scalas, Historical notes on the M-Wright function/ Mainardi function, *Commun. Appl. Ind. Math.*DOI: 10.1685/journal.caim.496, 1-6 (2014).
[9] M. El-Shahed and A. Salem, An extension of Wright fucnction and its properties, *J. Math.* Article ID 950728,11 pages (2015).
[10] M. Raza, M. U. Din and S. Nawaz Malik, Certain geometric properties of normalized Wright functions, *J. Funct. Spac.* 1–8 (2016).
[11] R. B. Paris and V. Vinogradov, Asymptotic and structural properties of special cases of the Wright function arising in probability theory, *Lithuanian Math. J.* **56(3)**, 377–409 (2016).
[12] K. Mehrez, Functional inequalites for the Wright fucntions, *Int. Transf Spec. Funct.* **28(2)**, 130–144 (2017).
[13] L. Moslehi and A. Ansari, On M- Wright transform and time-fractional diffusion equations, *Int. Trans. Spec. Fuct.* **28(2)**, 113–129 (2017).
[14] H. L. Krall and O. Frink, A new class of othogonal ploynomails, the Bessel polynomials, *Trans. Amer. Math. Soc.* **65**, 100–115 (1949).
[15] R. K. Singh and K. N. Srivastava, *A note on generalization of Laguerre and Humbert polynomials*, Ricerca(Napoli) **(2)**, 14 (1963), 11-21, Errata, ibid. **15**, 63 (1964).
[16] S. K. Raizada, *A study of unified representation of special function of mathematical physics and their use in statistical and boundary value problem*, Ph.D. Thesis, Bundelkhand Univ. Jhansi, India, 1991.
[17] M. Saigo, S. P. Goyal and S. Saxena, A theorem realting a generalized Wely fractional integral, Laplace and Verma transform with applications, *J. Fract. Calc.* **13**, 43–56 (1998).
[18] O. I. Marichev, *Handbook of integral transform and high transcendental functions*, Theory and Algorithmic Tables Ellis Horwood, Chichester, Johna Willey and Sons, New York, 1983.
[19] S. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integral and derivatives, theory and applications*, Gordon and Beach, Sci. Publ. New York, 1993.
[20] V. S. Kiryakova, *Generalized fractional calculus and applications*, Pitman Research Notes in Mathematics, 301, John Willey and Sons, New York, 1994.
[21] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics. In fractals and Fractional calculus in continuum mechanics (A. Carpintery and F. Mainardi, Editors) (Udine, 1996), CIAM Courses and Lectures **378**, 291–348 (1997).
[22] I. Podlubny, *Fractional differential equations*, Academic Press, New York, NY, USA, 1999.
[23] P. L. Butzer and U. Westphal, *An introduction to fractional calculus*. In: R. Hilfer, Ed. Applications of fractional calculus in physics, World Scientfic, 2000.
[24] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus: models and numerical method (Series on complexity, Nonlinearty and Chaos)*, World Scientific, Boston Mass, Vol. 3, 2012.
[25] D. Valerio, J. J. Trujillo, M. Rivero, J. A. T. Machado and D. Baleanu, Fractional calculus: A survey of useful formulas, *Eur. Phys. J. Spec. Top.* **8(222)**, 1827–1846 (2013).
[26] V. E. Tarasov, Leibniz rule and fractional derivative of power functions, *J. Comput. Nonlin. Dynam.* **11(3)**, 121–132 (2015).
[27] V. E. Tarasov, On chain rule for fractional derivatives, *Commun. Nonlin. Sci. Numer. Simul.* **30**, 1–4 (2016).
[28] Y. Zhang, H. G. Sun, H. H. Stowell, M. Zayernouri and S. E. Hansen, A review of applications of fractional calculus in earth system dynamics, *Chaos, Solit. Fract.* **102**, 29–46 (2017).

[29] H. M. Srivastava, Remarks on some families of fractional-order differential equations, *Int. Trans. Spec. Funct.* **28(7)**, 560–564 (2017).
[30] M. Saigo, A remark on integral operators involving the gauss hypergeometric funcion, *Math. Rep. Colleg. Gen. Kyushu Univ.* **11**, 135–143 (1978).
[31] H. M. Srivastava and R. K. Saxena, Operators of fractional integration and their applications, *Appl. Math. Comput.* **118(1)**, 1–52 (2001).
[32] V. S. Kiryakova, A brief story about the operators of the generalized fractional calculus, *Fract. Calc. Appl. Anal.* **11(2)**, 203–220 (2008).
[33] M. Saigo and N. Meada, More generalization of fractional calculus, Transform methods and special function, Varna Bulgaria, 386–400 (1996).