Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110607

On the Weighted Pseudo Almost Periodic Solutions of Nonlinear Functional Duffing Equation

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Received: 12 Jul. 2017, Revised: 18 Sep. 2017, Accepted: 22 Sep. 2017
Published online: 1 Nov. 2017

Abstract: This paper investigates a nonlinear Duffing equation with a variable time lag. By using a fixed point theorem, weighted pseudo almost periodic functions and differential inequality techniques, we establish new criteria for existence,uniqueness and globally exponential stability of weighted pseudo almost periodic solutions. An example with its numerical simulation is given to show the applicability of the proposed results. The results obtained is new and complements that found in the literature.

Keywords: Weighted pseudo almost periodic solution, Duffing equation, dichotomy, fixed point, variable time lag

1 Introduction

As we know Duffing equation represents the motion of a mechanical system in a twin-well potential field. Due to the applications of Duffing type equations in physics, mechanics and engineering technique fields, dynamic behaviors of that nonlinear equations have been investigated by many authors (see[3,5,6]). The authors of [5] considered the existence of almost periodic solutions for the following Duffing equation:

$$x'' + cx'(t) - ax(t) + bx^m(t - \tau(t)) = p(t). \qquad (1)$$

Let

$$y = x' + \xi x - \Phi_1(t),\ \Phi_2(t) = p(t) + (\xi - c)\Phi_1(t) - \Phi_1'(t).$$

$$\Phi_2(t) = p(t) + (\xi - c)\Phi_1(t) - \Phi_1'(t). \qquad (2)$$

By (2), equation (1) is transformed into the following system:

$$\begin{aligned} \frac{dx}{dt} &= -\xi x(t) + y(t) + \Phi_1(t), \qquad (3)\\ \frac{dy}{dt} &= -(c-\xi)y(t) + (a - \xi(\xi - c))x(t) \\ &\quad - bx^m(t-\tau(t)) + \Phi_2(t). \end{aligned}$$

Then, instead of constants, considering the coefficients in (1) as variable functions, the authors of [6] established conditions to guarantee the existence of almost periodic solutions of the following system:

$$\begin{aligned} \frac{dx}{dt} &= -\delta_1(t)x(t) + y(t) + \Phi_1(t), \qquad (4)\\ \frac{dy}{dt} &= \delta_2(t)y(t) + (\alpha(t) - \delta_2^2(t)x(t) - \eta(t)x^m(t-\tau(t)) \\ &\quad + \Phi_2(t). \end{aligned}$$

Liu and Tunç [3] gave some conditions for existence of pseudo almost periodic solutions of system (4). Here we introduce a new class of functions called weighted pseudo-almost periodic functions (see[1,2,4]) generalized almost periodic and pseudo almost periodic functions. The main idea is to put a weight(a locally integrable function on all R) on ergodic component appearing in (see[1,2]) and to obtain the weight space. In this way, a pseudo-almost periodic function appears as a perturbation of an almost-periodic function by an ergodic component of weight space.

Motivated by the mentioned papers, we study the existence, uniqueness and globally exponential stability of weighted pseudo almost periodic solutions of the following Duffing equation with a variable time lag $\tau(t)$:

$$x'' + c(t)x'(t) - a(t)x(t) + b(t)x^m(t-\tau(t)) = p(t), \qquad (5)$$

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where $t \in R$. Let $\delta_1(t)$ be a continuous differentiable function on R. Set

$$y(t) = x'(t) = -\alpha_1(t)x(t) + y(t). \tag{6}$$

Then we transform (5) into the following system:

$$\frac{dx}{dt} = -\alpha_1(t)x(t) + y(t), \tag{7}$$
$$\frac{dy}{dt} = -\alpha_2(t)y(t) + \eta(t)x(t) - b(t)x^m(t-\tau(t)) + p(t),$$

where $\eta(t) = a(t) + \alpha_1'(t) + \alpha_1(t)\alpha_2(t)$, $\alpha_2(t) = c(t) - \alpha_1(t)$. The initial condition for (7) is given by

$$x(s) = \phi_1(s), \quad y(s) = \phi_2(s), \quad s \in [-\tau^+, 0], \tag{8}$$

where $\phi_i \in C([-\tau^+, 0], R)$. To the best of our knowledge, up to now, there are no results available on the existence, uniqueness and globally exponential stability of weighted pseudo almost periodic solution for Duffing type equations. Our aim is to study these facts for weighted pseudo almost periodic solutions of (7). The proof is based on the properties of weighted pseudo almost periodic functions, exponential dichotomy and a fixed point theorem. Our results are new and complementary to the previously known results in the literature.

2 Preliminaries

Throughout this paper, we will use the following concepts and notations. $X = (x_1, x_2, ..., x_n) \in R^n$ denotes column vector. Define $|X| = (|x_1|, |x_2|, ..., |x_n|)^T$ and $\|X\| = \max_{1 \le i \le n} |x_i|$, in which the symbol T denotes transpose of a vector. For matrices or vectors U and V, $U \le V$ (resp. $U > V$) means that $U - V \ge 0$ (resp. $U - V > 0$). $BC(R, R^n)$ denotes the set of bounded and continuous functions from R to R, and $BUC(R, R)$ denotes the set of bounded and uniformly continuous functions from R to R.Note that $(BC(R, R^n), \|.\|_\infty)$ is a Banach space and $\|.\|_\infty$ denotes the supremum norm $\|f\|_\infty := \sup_{t \in R} \|f\|$. In this work,for a given a bounded continuous f defined on R, let f^+ and f_- be defined as

$$f^+ = \sup_{t \in R} |f(t)|, \quad f_- = \inf_{t \in R} |f(t)|.$$

Let U denotes the collection of weight functions $\rho : R \to (0, \infty)$ which are locally integrable over R such that $\rho > 0$ almost everywhere. If $\rho \in U$, then we set

$$\mu(T, \rho) = \int_{-T}^{T} \rho(x)dx, \quad T > 0.$$

In the particular case, when $\rho(x) = 1$ for each $x \in R$, we are exclusively interested in those weights ρ, for which $\lim_{T \to \infty} \mu(T, \rho) = \infty$.

Let $U_\infty := \{\rho \in U : \lim_{T \to \infty} \mu(T, \rho) = \infty\}$ and

$$U_B := \{\rho \in U : \rho \ \texttt{ is bounded with } \ \inf_{t \in R} \rho(x) > 0\}.$$

Definition 2.1.[1] A function $f : R \to X$ is said to be almost periodic, if for any $\varepsilon > 0$, there is a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$ there exists τ such that the inequality

$$\|f(t+\tau) - f(t)\| < \varepsilon$$

is satisfied for all $t \in R$. The number τ is called an ε-translation number of $f(t)$.

To introduce weighted pseudo almost periodic functions, we need to define the weighted ergodic space $PAP_0(R, R^n, \rho)$. Weighted pseudo almost periodic functions will then appear as perturbations of almost periodic functions by elements of $PAP_0(R, R^n, \rho)$. Let $\rho \in U_\infty$. Define

$$PAP_0(R, R^n, \rho) = \{f \in BC(R, R^n) :$$
$$\lim_{T \to \infty} \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \|f(x)\| \rho(x)dx = 0\}.$$

Definition 2.2.[1] Let $\rho \in U_\infty$. A function $f \in BC(R, R^n)$ is called weighted pseudo almost periodic if it can be expressed as $f = g + \varphi$, where $g \in AP(R, R^n)$ and $\varphi \in PAP_0(R, R^n, \rho)$. The collection of such functions will be denoted by $PAP(R, R^n, \rho)$.

Lemma 2.1.[7] Let $\rho : R \to (0, \infty), \rho \in U_\infty$ be a continuous function and assume that

$$\sup_{t \in R} [\frac{\rho(t+r)}{\rho(t)}] < \infty$$

and

$$\sup_{T > 0} [\frac{\mu(T+r, \rho)}{\mu(T, \rho)}] < \infty \ \texttt{for each} \quad t \in R.$$

If $\varphi(.) \in PAP(R, R, \rho)$, then $\varphi(. - h) \in PAP(R, R, \rho)$.

Lemma 2.1.[7] If $\varphi, \psi \in PAP(R, R, \rho)$, then $\varphi \times \psi \in PAP(R, R, \rho)$.

Lemma 2.2.[8] If $f(t) \in PAP(R, R, \rho)$, $\tau(t) \in C^1(R, R)$ and $\tau(t) \ge 0, \tau'(t) \le 1$, then $f(t - \tau(t)) \in PAP(R, R, \rho)$.

Definition 2.3. Let $\nu_*(t) = (\upsilon_*(t), \rho_*(t))^T$ be an weighted pseudo almost periodic solution of (7) with initial value $\phi_*(t) = (\phi_*(t), \phi_*(t))^T$. If there exist positive constants λ and $M > 1$ such that for any arbitrary solution $\nu(t) = (\upsilon(t), \rho(t))^T$ of (7) with initial value $\phi(t) = (\phi_1(t), \phi_2(t))^T$ satisfies

$$|\nu(t) - \nu_*(t)|_1 \le Me^{-\lambda(t-t_0)}$$
$$\times \|\phi(t) - \phi_*(t)\|, \forall t, t_0 \in [-\tau^+, +\infty), t \ge t_0$$

where, $|\nu(t)|_1 = \max\{|\upsilon(t) - \upsilon_*(t)|, \rho(t) - \rho_*(t)\}$, $\|\phi(t) - \phi_*(t)\| = \max\{|\phi_1(t) - \phi_{*1}(t)|, |\phi_2(t) - \phi_{*2}(t)|\}$, and $|\phi - \phi_*|_0 = \sup_{t \in [-\tau^+, 0]} |\phi_i - \phi_{*i}|$, $(i = 1, 2)$, then the solution $\nu_*(t)$ is said to be globally exponential stable.

3 Existence of solutions

Assumptions: Through the paper we suppose that the following conditions hold.

(A1) $m > 1$, m is a integer, $\alpha_1(t),\alpha_2(t) \in AP(R,R^+), b(t),\eta(t),p(t) \in PAP(R,R^+,\rho)$ and $\eta(t) \neq 0$, for $t \in R$.

(A2) $\rho : R \to (0,\infty), \rho \in U_\infty$ is continuous and

$$\sup_{t\in R}[\frac{\rho(t+r)}{\rho(t)}] < \infty, \quad \sup_{T>0}[\frac{\mu(T+r,\rho)}{\mu(T,\rho)}] < \infty.$$

(A3) Let $\lambda,\alpha_1^*,\alpha_2^*,\alpha^*,k,\sigma$ and r be constants such that

$$\alpha_1^* = \inf_{t\in R}|\alpha_1(t)|, \quad \alpha_2^* = \inf_{t\in R}|\alpha_2(t)|, \quad \alpha^* = \min\{\alpha_1^*,\alpha_2^*\},$$

$$0 < \lambda < \delta^*, k = \frac{\sup_{t\in R}|p(t)|}{\alpha^*},$$

$$\sigma = \max\{\frac{1}{\alpha^*},\frac{\eta^+ + b^+}{\alpha^*}\},$$

$$r = \max\left\{\frac{1}{\alpha^*},\frac{\eta^+ + mb^+(\frac{2k}{1-\sigma})^{m-1}e^{\lambda\tau^+}}{\alpha^*}\right\}.$$

(A4)

$$\sup_{T>0}\{\int_{-T}^{T} e^{-\alpha^*(T+t)}\rho(t)dt\} < \infty.$$

Lemma 3.1. Suppose that assumptions (A1)-(A4) hold. Define a nonlinear operator G for each $\phi = (\phi_1,\phi_2) \in PAP(R,R^2,\rho), (G\phi) := x_\phi(t)$, where

$$x_\phi(t)) = \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t \alpha_1(w)dw}\phi_2(s)ds \\ -\int_t^{+\infty} e^{-\int_s^t \alpha_2(w)dw}(\eta(s)\phi_1(s)-b(s)\phi_1^m(s-\tau(s)))ds \end{pmatrix}.$$

Then $G\phi \in PAP(R,R^2,\rho)$.

The proof of this lemma is obvious. Therefore, we omit the details of the proof.

Theorem 3.1. Let assumptions (A1)-(A4) be hold and

$$\sigma < 1, \qquad \frac{k}{1-\sigma} < 1, \qquad r < 1.$$

Then there exists a unique weighted pseudo almost periodic solution of system (7) in the region

$$S = \{\phi | \|\phi-\phi_0\| \leq \frac{\sigma k}{1-\sigma}, \phi \in PAP(R,R^2,\rho)\},$$

where

$$\phi_0 = (0,\int_t^{+\infty} e^{-\int_t^s \alpha_2(w)dw}p(s)ds).$$

Proof. Define a mapping $\Omega : S \to S$, by setting

$$(\Omega\phi)(t) = \begin{pmatrix} x_\phi \\ y_\phi \end{pmatrix},$$

where for all $\phi \in S$,

$$x_\phi = \int_{-\infty}^{t} e^{-\int_s^t \alpha_1(w)dw}\phi_2(s))ds,$$

$$y_\phi = -\int_t^{+\infty} e^{-\int_s^t \alpha_2(w)dw}(\eta(s)\phi_1(s)-b(s)\phi_1^m(s-\tau(s)))ds.$$

It is clear that

$$\begin{aligned}\|\phi_0\|_\infty &\leq \sup_{t\in R}\max\left\{0,\int_t^{+\infty} e^{-\int_t^s \alpha_2(w)dw}p(s)ds\right\}\\ &\leq \frac{1}{\alpha^*}\max\{0,\sup_{t\in R}|p(t)|\} = k.\end{aligned}$$

Therefore, for any $\phi \in S$, we have

$$\|\phi\|_\infty \leq \|\phi-\phi_0\|_\infty + \|\phi_0\|_\infty \leq \frac{\sigma k}{1-\sigma} + k = \frac{k}{1-\sigma} < 1.$$

Hence, it follows that

$$\begin{aligned}\|\Omega\phi-\phi_0\|_\infty &\leq \sup_{t\in R}\max\{|\int_{-\infty}^{t} e^{-\int_s^t \alpha_1(w)dw}\phi_2(s)(s)ds|,\\ &\quad |\int_{-\infty}^{t} e^{-\int_s^t \alpha_2(w)dw}(\eta(s)\phi_1(s)\\ &\quad -b(s)\phi_1^m(s-\tau(s)))ds|\}\\ &\leq \sup_{t\in R}\max\{\int_{-\infty}^{t} e^{-\int_s^t \alpha_1(w)dw}ds\|\phi\|,\\ &\quad \sup_{t\in R}[\int_{-\infty}^{t} e^{-\int_s^t \alpha_2(w)dw}ds(\eta^+\|\phi\| + b^+|\phi\|^m)\}\\ &\leq \max\{\frac{1}{\alpha^*},\frac{\eta^+ + b^+}{\alpha^*}\}\|\phi\|\\ &\leq \frac{\sigma k}{1-\sigma}.\end{aligned}$$

So, the mapping Ω is a self-mapping from S to S. In addition, for $\phi,\psi \in S$, we get

$$\begin{aligned}&|(\Omega(\phi(t))-\Omega(\psi(t)))| = (|(\Omega(\phi(t))-\Omega(\psi(t)))_1|,\\ &|(\Omega(\phi(t))-\Omega(\psi(t)))_2|)^T\\ &= (|\int_{-\infty}^{t} e^{-\int_s^t \alpha_1(w)dw}(\phi_2(s)-\psi_2(s))ds|,|\int_t^{+\infty} e^{-\int_s^t \alpha_2(w)dw}\\ &\times(\eta(s)(\phi_1(s)-\psi_1(s))-b(s)(\phi_1^m(s-\tau(s))\\ &-\psi_1^m(s-\tau(s))))ds|)^T\\ &= (|\int_{-\infty}^{t} e^{-\int_s^t \alpha_1(w)dw}(\phi_2(s)-\psi(s))ds|,|\int_t^{+\infty} e^{-\int_s^t \alpha_2(w)dw}\\ &\times(\eta(s)(\phi_1(s)-\psi_1(s))-b(s)m(\psi_1(s-\tau(s))\\ &+h(s)((\phi_1(s-\tau(s))(s)-\psi_1(s-\tau(s))))^{m-1}\\ &\times(\phi_1(s-\tau(s))-\psi_1(s-\tau(s)))|))^T,\end{aligned}$$

where $h(s) \in (0,1)$. Then,

$$
\begin{aligned}
&|(\Omega(\phi(t)) - \Omega(\psi(t)))| \\
&\leq \Big(\int_{-\infty}^{t} e^{-\int_s^t \alpha_1(w)dw} ds \sup_{s\in R} |\phi_2(s) - \psi_2(s)|, \\
&\int_t^{+\infty} e^{-\int_s^t \alpha_2(w)dw} |\eta(s)(\phi_1(s) - \psi_1(s)) \\
&- b(s)m\psi_1(s-\tau(s)) + h(s)(\phi_1(s-\tau(s)) \\
&- \psi_1(s-\tau(s)))^{m-1}(\phi_1(s-\tau(s)) - \psi_1(s-\tau(s))|)\Big)^T \\
&\leq \Big(\frac{1}{\alpha^*}\|\phi-\psi\|_\infty, \int_t^{+\infty} e^{-\int_s^t \alpha_2(w)dw} \\
&\times |\eta(s)(\phi_1(s) - \psi_1(s)) - b(s)m((1-h(s))\psi_1(s-\tau(s)) \\
&+ h(s)(\phi_1(s-\tau(s)) - \psi_1(s-\tau(s))))^{m-1} \\
&\times (\phi_1(s-\tau(s)) - \psi_1(s-\tau(s))|)\Big)^T
\end{aligned}
$$

so that

$$
\begin{aligned}
&\|(\Omega(\phi(t)) - \Omega(\psi(t))\| \\
&\leq \max\left\{ \frac{1}{\alpha^*}, \frac{\eta^+ + mb^+(\frac{2k}{1-\sigma})^{m-1}}{\alpha^*} \right\} \|\phi-\psi\|_\infty \\
&\leq \max\left\{ \frac{1}{\alpha^*}, \frac{\eta^+ + mb^+(\frac{2k}{1-\sigma})^{m-1}e^{\lambda\tau^+}}{\alpha^*} \right\} \|\phi-\psi\|_\infty \\
&= r\|\phi-\psi\|_\infty.
\end{aligned}
$$

It follows from $r < 1$ that $\Omega : S \to S$ is a contraction mapping. By the fixed point theorem, Ω have a unique fixed point $\phi^* \in S$ such that $\Omega\phi^* = \phi^*$. ϕ^* satisfies (7). So ϕ^* is a weighted pseudo almost periodic solution of (7). Hence we can conclude to the end of the proof of Theorem 3.1.

4 Exponential stable of solutions

Theorem 4.1. Let assumptions (A1)-(A4) be hold. Then, the weighted pseudo almost periodic solution of system (7) is globally exponential stable.

Proof. By Theorem 3.1, (7) has a weighted pseudo almost periodic solution $\nu_*(t) = (\upsilon_*(t), \rho_*(t))^T \in S$ with the initial value $\phi_* = (\phi_{*1}(t), \phi_{*2}(t))^T$. Suppose that $\nu(t) = (\upsilon(t), \rho(t))^T \in S$ is an arbitrary solution of (7) with the initial value $\phi(s) = (\phi_1(s), \phi_2(s))^T$. Denote $w(t) = (u(t), v(t))^T$, where $u(t) = x(t) - x_*(t)$, $v(t) = y(t) - y_*(t)$. Then it follows from (7) that

$$
\begin{cases}
u'(t) = -\alpha_1(t)u(t) + v(t), \\
v'(t) = -\alpha_2(t)v(t) + \eta(t)u(t) \\
-b(t)(x^m(t-\tau(t)) - x_*^m(t-\tau(t))).
\end{cases} \tag{9}
$$

Multiplying both side of equation of (9) by $e^{-\int_{t_0}^t \alpha_1(s)ds}$ and $e^{-\int_{t_0}^t \alpha_2(s)ds}$, respectively, then integrating the obtained estimate on $[t_0, t]$, where $t_0 \in [-\tau^+, 0]$, we get

$$
u(t) = \phi_1(t_0)e^{-\int_{t_0}^t \alpha_1(s)ds} + \int_{t_0}^t e^{-\int_s^t \alpha_1(w)dw} v(s)ds, \tag{10}
$$

$$
\begin{aligned}
v(t) &= \phi_2(t_0)e^{-\int_{t_0}^t \alpha_2(s)ds} + \int_{t_0}^t e^{-\int_s^t \alpha_2(w)dw} \\
&\times (\eta(s)u(s) - b(s)(x^m(s-\tau(s)) - x_*^m(s-\tau(s))))ds.
\end{aligned}
$$

Since $0 < \lambda < \min\{\inf|\alpha_1(t)|, \inf|\alpha_2(t)|\}$, it is clear that

$$
\begin{aligned}
|\nu(t) - \nu_*(t)|_1 = |\phi(t)| \leq \|\phi\| \\
\leq Me^{-\lambda(t-t_0)}\|\phi\|, \ \forall t \in [-\tau^+, t_0].
\end{aligned} \tag{11}
$$

We claim that

$$
|\nu(t) - \nu_*(t)|_1 \leq Me^{-\lambda(t-t_0}\|\phi\|, \quad \forall t \in (t_0, +\infty). \tag{12}
$$

To prove this claim, we show that for any constant $p > 1$, the following inequality holds:

$$
|\nu(t) - \nu_*(t)|_1 \leq pMe^{-\lambda(t-t_0)}\|\phi\|, \quad \forall t \in (t_0, +\infty), \tag{13}
$$

which implies that

$$
|u(t)| = |x(t) - x_*(t)| \leq Mpe^{-\lambda(t-t_0)}\|\phi\|, \quad \forall t \in (t_0, +\infty) \tag{14}
$$

and

$$
|v(t)| = |y(t) - y_*(t)| \leq Mpe^{-\lambda(t-t_0)}\|\phi\|, \quad \forall t \in (t_0, +\infty). \tag{15}
$$

By the way of contradiction, assume that (13) does not hold. We will have the following three cases.

Case 1. Inequality (15) is true, but (14) is not true. Then there exist $t_1 \in (t_0, +\infty)$ and $\alpha \geq 1$ such that

$$
|u(t_1)| = \Delta Mpe^{-\lambda(t_1-t_0)}\|\phi\|, \tag{16}
$$

$$
|u(t)| < \Delta Mpe^{-\lambda(t-t_0)}\|\phi\|, \quad \forall t \in (t_0, t_1).
$$

Note that, in view of (10), we have

$$
\begin{aligned}
|u(t_1)| &= |\phi_1(t_0)e^{-\int_{t_0}^{t_1} \alpha_1(s)ds} + \int_{t_0}^{t_1} e^{-\int_s^{t_1} \alpha_1(w)dw} v(s)ds| \\
&\leq e^{-\alpha^*(t_1-t_0)}\|\phi\| + \int_{t_0}^{t_1} e^{-\alpha^*(t_1-s)}\Delta Mpe^{-\lambda(s-t_0)}\|\phi\|ds \\
&\leq e^{-\lambda(t_1-t_0)}\|\phi\| + \Delta Mp\|\phi\| \int_{t_0}^{t_1} e^{-\alpha^*(t_1-s)}e^{-\lambda(s-t_0)}ds \\
&\leq e^{-\lambda(t_1-t_0)}\|\phi\| + \Delta Mp\|\phi\| \int_{t_0}^{t_1} e^{-\alpha^*(t_1-s)}ds
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-\lambda(t_1-t_0)}\|\phi\| - \frac{1}{\alpha^*}\Delta Mp\|\phi\|(1 - e^{-\alpha^*(t_1-t_0)}) \qquad (17) \\
&= \Delta Mpe^{-\lambda(t_1-t_0)}\|\phi\|\Big(\frac{1}{\Delta M} - \frac{1}{\alpha^*}\Big) \\
&< \Delta Mpe^{-\lambda(t_1-t_0)}\|\phi\|\Big(\frac{1}{\Delta M} + \frac{1}{\alpha^*}\Big) \\
&< \Delta Mpe^{-\lambda(t_1-t_0)}\|\phi\|.
\end{aligned}
$$

Thus we get a contradiction.

Case 2. Inequality (14) is true, but (15) is not true. Then there exist $t_2 \in (t_0, +\infty)$ and $\Delta \geq 1$ such that

$$|v(t_2)| = \Delta M p e^{-\lambda(t_2-t_0)}\|\phi\|, \quad (18)$$

$$|v(t)| < \Delta M p e^{-\lambda(t-t_0)}\|\phi\|, \quad \forall t \in (t_0, t_2).$$

Note that, in view of (10), we have

$$\begin{aligned}
&|v(t_2)| = |\phi_2(t_0) e^{-\int_{t_0}^{t_2}\alpha_2(s)ds} \int_{t_0}^{t_2} e^{-\int_s^{t_2}\alpha_2(w)dw} \\
&\times (\eta(s)u(s) - b(s)(x^m(s-\tau(s)) - x_*^m(s-\tau(s))))ds| \\
&\leq e^{-\int_{t_0}^{t_2}\alpha_2(s)ds}\|\phi\| + \int_{t_0}^{t_2} e^{-\alpha^*(t_2-s)} \\
&\times |(\eta(s)u(s) - b(s)(x^m(s-\tau(s)) - x_*^m(s-\tau(s)))d)s| \\
&\leq e^{-\int_{t_0}^{t_2}\alpha_2(s)ds}\|\phi\| + \int_{t_0}^{t_2} e^{-\alpha^*(t_2-s)} \\
&\times |(\eta(s)u(s) - b(s)m(x_*^m(s-\tau(s)) + \varsigma(s)(x(s-\tau(s)) \\
&- x_*(s-\tau(s))))^{m-1}x(s-\tau(s)) - x_*(s-\tau(s))|)ds \\
&= e^{-\int_{t_0}^{t_2}\alpha_2(s)ds}\|\phi\| + \int_{t_0}^{t_2} e^{-\alpha^*(t_2-s)} \\
&\times |(\eta(s)u(s) - b(s)m((1-\varsigma(s))x_*^m(s-\tau(s)) \\
&+ \varsigma(s)(x(s-\tau(s))))^{m-1}x(s-\tau(s)) - x_*(s-\tau(s))|)ds \\
&\leq e^{-\alpha^*(t_2-t_0)}\|\phi\| + \int_{t_0}^{t_2} e^{-\alpha^*(t_2-s)}(\eta^+|u(s)| \\
&+ b^+m|((1-\varsigma(s))x_*^m(s-\tau(s)) + \varsigma(s)(x(s-\tau(s))))^{m-1} \\
&\times x(s-\tau(s)) - x_*(s-\tau(s))|)ds \\
&\leq e^{-\alpha^*(t_2-t_0)}\|\phi\| + \int_{t_0}^{t_2} e^{-\alpha^*(t_2-s)}(\eta^+|u(s)| \\
&+ b^+m(\frac{2k}{1-\sigma})^{m-1}|u(s-\tau(s))|)ds \\
&\leq e^{-\alpha^*(t_2-s)}\|\phi\| + \eta^+\Delta_1 p M e^{-\lambda(t_2-t_0)}\|\phi\| \int_{t_0}^{t_2} e^{-\alpha^*(t_2-s)}ds \\
&+ b^+m(\frac{2k}{1-\sigma})^{m-1}\Delta_1 p M e^{-\lambda(t_2-t_0)} \\
&\times \|\phi\| \int_{t_0}^{t_2} e^{-\alpha^*(t_2-s)} e^{-\lambda(s-\tau(s)-t_2)}ds \\
&\leq e^{-\alpha^*(t_2-t_0)}\|\phi\| + \frac{1}{-\alpha^*}\eta^+\Delta_1 p M e^{-\lambda(t_2-t_0)} \\
&\times \|\phi\| \int_{t_0}^{t_2} e^{-\alpha^*(t_2-s)}ds \\
&+ \frac{1}{-\alpha^*} b^+m(\frac{2k}{1-\sigma})^{m-1}\Delta_1 p M e^{-\lambda(t_2-t_0)} \\
&\times \|\phi\| \exp\{\lambda\tau^+\} \int_{t_0}^{t_2} e^{-\alpha^*(t_2-s)}ds \\
&\leq e^{-\alpha^*(t_2-t_0)}\|\phi\| - \frac{1}{\alpha^*}\eta^+\Delta_1 p M e^{-\lambda(t_2-t_0)} \\
&\times \|\phi\|(e^{-\alpha^*(t_2-t_0)} - 1) - \frac{1}{\alpha^*} b^+m(\frac{2k}{1-\sigma})^{m-1}\Delta_1 p M e^{-\lambda(t_2-t} \\
&\times \|\phi\| \exp\{\lambda\tau^+\}(e^{-\alpha^*(t_2-t_0)} - 1) \\
&\leq \Delta_1 p M e^{-\lambda(t_2-t_0)}\|\phi\|(\frac{1}{\Delta_1 M} + \frac{\eta^+ + mb^+(\frac{2k}{1-\sigma})^{m-1}e^{\lambda\tau^+}}{\alpha^*}) \\
&< \Delta_1 p M e^{-\lambda(t_2-t_0)}\|\phi\|,
\end{aligned}$$

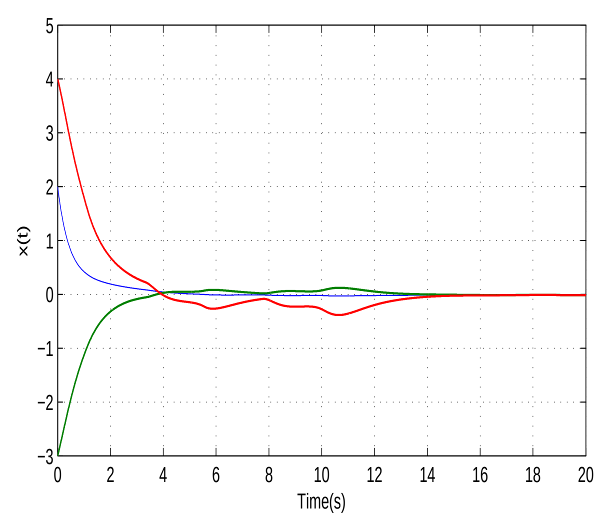
where $0 \leq \varsigma(s) \leq 1$. We also get contradiction.

Case 3. Both of inequalities (14) and (15) are wrong. By Case 1 and Case 2, we can obtain a contraction. Therefore, (13) holds. Let $p \to 1$. Then (12) holds. Hence, the weighted pseudo almost periodic solution $v_*(t)$ of (7) is globally exponential stable.

Example. As a special case of (5), consider the following second order differential equation;

$$\begin{aligned}
x'' &+ (3.1 + 0.01\sin\sqrt{2}t)x' + (2 + 0.01\cos t)x \\
&+ (0.03 + 0.02\cos\sqrt{2}t)x^2(t - \cos\sqrt{2}t) \\
&= 0.03 - 0.01\cos\sqrt{2}t + e^{-t}.
\end{aligned} \quad (19)$$

It is easy to see that $c(t) = 3.1 + 0.01\sin\sqrt{2}t$, $a(t) = -2 - 0.01\cos t$, $b(t) = 0.03 + 0.02\cos\sqrt{2}t$, $p(t) = 0.03 - 0.01\sin\sqrt{3}t + e^{-t}$, $\tau(t) = \cos\sqrt{2}t, m = 3$. If we set $\alpha_1(t) = 1.6 + 0.01\sin t$, then we have $\alpha_2(t) = 1.5$, $\eta(t) = 0.4 + 0.015\sin t, \rho(t) = e^t$ for all $t \geq 0$, $\rho(t) = 1$ for all $t < 0$. Therefore, we have that $\sigma \approx 0.43 < 1$, $r = 0.4$, $k/(1-0.57) \approx 0.081 < 1$, $\sup_{T>0}\{\int_{-T}^{T} e^{-\alpha^*(T+t)}\rho(t)dt\} < \infty$, which imply that all conditions of Theorem 4.1 are satisfied. Hence, (19) has an weighted pseudo almost periodic solution, which is globally exponential stable.



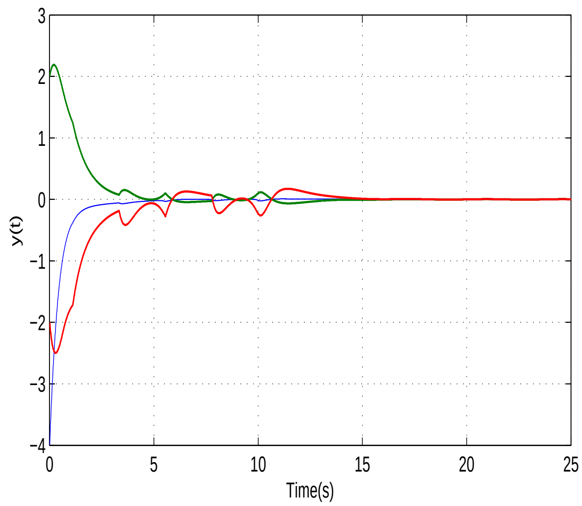


Fig. 1: Numeric solutions $(x(t),y(t))$ of system (7) for initial values $\phi(s) \equiv (4,-2)^T,(2,-4)^T,(-3,2)^T, s \in [-1,0]$

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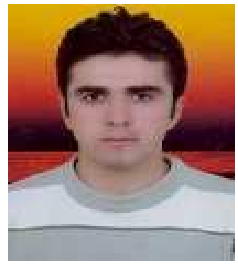

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