Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110606

Second Order Fuzzy Fractional Differential Equations Under Caputo's H-Differentiability

Shatha Hasan[1], *Ahmad Alawneh*[1], *Mohammad Al-Momani*[2] *and Shaher Momani*[1,3,*]

[1] Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
[2] Department of Management Information Systems, Faculty of Economics and Administrative Science, Zarqa University, Zarqa, Jordan
[3] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah, 21589, Kingdom of Saudi Arabia

Received: 1 Aug. 2017, Revised: 2 Sep. 2017, Accepted: 6 Sep. 2017
Published online: 1 Nov. 2017

Abstract: The aim of this paper is to use the concept of the generalized H-derivative to define fuzzy Caputo's H-derivative of order $\beta \in (1,2]$. Our definition is an extension of fuzzy Caputo's H-derivative of order $\beta \in (0,1]$ and higher order H-derivative of integer order. After that, we study fuzzy fractional initial value problems of order $\beta \in (1,2]$ and give an algorithm to solve them based on the characterization theorem. Finally, we apply the reproducing kernel Hilbert space method to obtain approximate solutions of second order fuzzy fractional initial value problems and give some numerical examples.

Keywords: Caputo's H-derivative, second order fuzzy fractional differential equation, reproducing kernel Hilbert space

1 Introduction

Fractional calculus has recently attracted the attention of many researchers for its considerable importance in science [6,34]. But in many cases of modeling real world phenomena, information about the behavior of a dynamical system is uncertain. So fuzzy set theory was established by Zadeh in 1965 [7,8]. In 1978, Dubois and Prade introduced the notion of fuzzy real numbers and established some of their basic properties [9]. The term ”fuzzy differential equations” was coined in the same year by Kandel and Byatt [10]. Many definitions were suggested for a fuzzy derivative and then for studying fuzzy differential equations [11,12,13,14,15]. The most popular approach is using Hukuhara derivative [12,16].

Recently, the concept of fuzzy fractional differential equations (FFDEs) was introduced to consider a new type of dynamical systems [17]. In [18], the authors considered a generalization of the H-differentiability for the fractional case. In the last few years, several research works have been devoted to study and solve FFDEs of order $\beta \in (0,1]$, see [19,20,21,22,23,24,25,26,27].

In [3], a generalized concept of higher order H-derivative for fuzzy functions was introduced for integer order. Here, using the concept of generalized H-derivative, we define fuzzy Caputo's H-derivative of order $\beta \in (1,2]$ and solve second order fuzzy fractional initial value problems (FFIVPs) based on the characterization theorems [4,5,6]. We apply a modified reproducing kernel Hilbert space method (RKHSM) to obtain numerical solutions. To see some applications of the RKHSM for solving differential equations of different types, the reader is asked to refer to [28,29,30,31,32,35,36,37].

This paper is organized as follows: In section 2, we introduce some basics of fuzzy calculus and fractional calculus. In section 3, we define second order Caputo's H-derivative and prove some related results. An algorithm to solve second order FFDEs is given in section 4. Section 5 is devoted to apply a modified RKHSM to solve FFIVPs. This paper ends in section 6 with a conclusion.

2 Some Basics of Fuzzy Calculus and Fractional Calculus

In this section, we introduce some necessary definitions of fuzzy and fractional calculus.

Definition 2.1.[7] *A fuzzy set A in a universal set X is characterized by a membership function $u(x)$ which*

* Corresponding author e-mail: s.momani@ju.edu.jo

associates with each point in X a real number in the interval $[0,1]$.

Its r-cut representation is given by $[u]^r = \{x \in X : u(x) \geq r\}$ for $r \in (0,1]$ and $[u]^0 = \overline{\{x \in X : u(x) > 0\}}$. $[u]^0$ is called the support of A . A is normal if there is $x \in X$ with $u(x) = 1$. The core of A is core$(A) = \{x \in X : u(x) = 1\}$. A convex set A is a fuzzy convex set iff $u(\gamma x + (1-\gamma)y) \geq \min(u(x), u(y))$ for all $x, y \in X$ and $\gamma \in [0,1]$. If we take X to be the set of all real numbers $\Re$, then a special class of fuzzy sets results which is called the set of fuzzy numbers $\Re_F$. The following theorem gives the conditions that must be satisfied by two real valued functions u_1, u_2 defined on $[0,1]$ so that $[u_1(r), u_2(r)]$ is the parameterization form of a fuzzy number u for each $r \in [0,1]$.

Theorem 2.1.*[39] Suppose that $u_1, u_2 : [0,1] \to \Re$ satisfy the following conditions:*

1. *u_1 is a bounded monotonic nondecreasing left continuous function $\forall r \in (0,1]$ and right continuous for r=0.*
2. *u_2 is a bounded monotonic nonincreasing left continuous function $\forall r \in (0,1]$ and right continuous for r=0.*
3. *$u_1(1) \leq u_2(1)$ (which implies that $u_1(r) \leq u_2(r) \forall r \in [0,1]$).*

Then $u : \Re \to [0,1]$ which is defined by $u(x) = \sup\{r : u_1(r) \leq x \leq u_2(r)\}$ is a fuzzy number with parameterization $[u]^r = [u_1(r), u_2(r)]$. Moreover, if u is a fuzzy number with $[u]^r = [u_1(r), u_2(r)]$ (or simply, $[u_{1r}, u_{2r}]$), then the functions $u_1, u_2 : [0,1] \to \Re$ satisfy the conditions (1-3).

Addition and scalar multiplication in $\Re_F$ can be defined as those on intervals of $\Re$. So for any $\lambda \in \Re - \{0\}$, and $u, v \in \Re_F$ with $[u]^r = [u_{1r}, u_{2r}]$ and $[v]^r = [v_{1r}, v_{2r}]$, we have $[u+v]^r = [u]^r + [v]^r = [u_{1r} + v_{1r}, u_{2r} + v_{2r}]$, and $[\lambda u]^r = \lambda [u]^r = [\min\{\lambda u_{1r}, \lambda u_{2r}\}, \max\{\lambda u_{1r}, \lambda u_{2r}\}]$. While for subtraction, we use the H-difference, see [16]. The H-difference of $u, v \in \Re_F$, denoted by $u \ominus v = w$, is the fuzzy number that satisfies $u = v + w$. Its r-cut representation is $[u \ominus v]^r = [u_{1r} - v_{1r}, u_{2r} - v_{2r}]$.

Definition 2.2.*[40]The Housdorff metric D on $\Re_F$ is defined by $D : \Re_F \times \Re_F \to \Re^+ \cup \{0\}$ such that $D(u,v) = Sup_{r \in [0,1]} max\{|u_{1r} - v_{1r}|, |u_{2r} - v_{2r}|\}$ for any fuzzy numbers $u = (u_1, u_2)$ and $v = (v_1, v_2)$.*

A fuzzy function on an interval T is a mapping $F : T \to \Re_F$. If for fixed $t_0 \in T$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $|t - t_0| < \delta \Rightarrow D(F(t), F(t_0)) < \varepsilon$, then we say that F is continuous at t_0. If F is continuous $\forall t \in T$, then F is continuous on T [41]. A natural way for extending a crisp mapping $f : \Re \to \Re$ to a mapping $F : \Re_F \to \Re_F$ is Zadeh's extension principle [8]. Nguyen theorem gives a sufficient condition for when Zadeh's extension of a real valued function $f : \Re \times \Re \to \Re$, say $F : \Re_F \times \Re_F \to \Re_F$, is a well-defined fuzzy function.

Theorem 2.2.*[42] If $f : \Re \times \Re \to \Re$, is continuous, then $F : \Re_F \times \Re_F \to \Re_F$ is a well-defined function with r-cuts $[F(u,v)]^r = f([u]^r, [v]^r) = \{f(x,y) : x \in [u]^r, y \in [v]^r\} \forall r \in [0,1]$ and $u, v \in \Re_F$.*

For the differentiation of a fuzzy function, we use the concept of strongly generalized derivative [1]. It was given in 2005 as a generalization of the H-derivative introduced by Hukuhara in 1967 for set valued mappings and extended by Puri and Ralescu in 1983 for fuzzy valued mappings [12].

Definition 2.3.*[1]Let $F : (a,b) \to \Re_F$ and $t_0 \in (a,b)$. We say that F is strongly generalized differentiable at t_0 if there exists a fuzzy number $F'(t_0)$ such that*
(1) There exist $F(t_0+h) \ominus F(t_0)$ and $F(t_0) \ominus F(t_0-h)$ and
$$\lim_{h\to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0)$$
or
(2) There exist $F(t_0) \ominus F(t_0+h)$ and $F(t_0-h) \ominus F(t_0)$ and
$$\lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0+h)}{-h} = \lim_{h\to 0^+} \frac{F(t_0-h) \ominus F(t_0)}{-h} = F'(t_0).$$
The limits here are taken in the metric space $(\Re_F, D)$.

We say that F is (n)-differentiable for $n = 1,2$ if F is strongly generalized differentiable in the nth form and denote the (n)-derivative of F at t_0 by $F'(t_0) = D^1_n F(t_0)$. However, if $D^1_1 F(t_0)$ exists, then $D^1_2 F(t_0)$ doesn't exist [5].
Remark: In [1], the authors suggested four cases for the generalized H-derivative and proved that two of them are reduced to a crisp element. So, they are missing here.

Theorem 2.3.*[43]Let $F : [a,b] \to \Re_F$ be a strongly generalized differentiable function at $t_0 \in [a,b]$. Then:*
a) If F is (1)-differentiable at t_0, then F_{1r} and F_{2r} are differentiable at t_0 and $[F'(t_0)]^r = [F'_{1r}(t_0), F'_{2r}(t_0)], \forall r \in [0,1]$
b) If F is (2)-differentiable at t_0, then F_{1r} and F_{2r} are differentiable at t_0 and $[F'(t_0)]^r = [F'_{2r}(t_0), F'_{1r}(t_0)], \forall r \in [0,1]$

Based on definition 2.3, we have two possibilities to obtain the first order fuzzy derivative of a fuzzy function F. Consequently, there are four possibilities for the second fuzzy derivative which is defined as follows.

Definition 2.4.*[3]Let $F : (a,b) \to \Re_F$. We say that F is (n,m)-differentiable at $t_0 \in (a,b)$ if $F(t)$ is (n)-differentiable on a neighborhood of t_0 as a fuzzy function, and $F'(t)$ is (m)-differentiable at t_0. The second derivatives of F at t are denoted by $F''(t) = D^2_{n,m} F(t), n, m \in \{1,2\}$.*

Theorem 2.4.*[3]Let $D^1_1 F, D^1_2 F : (a,b) \to \Re_F$ be fuzzy functions with $[F(t)]^r = [F_{1r}(t), F_{2r}(t)], r \in [0,1]$.*
a) If $D^1_1 F$ is (1)-differentiable, then F'_{1r} and F'_{2r} are differentiable functions and $[D^2_{1,1} F(t)]^r = [F''_{1r}(t), F''_{2r}(t)]$.

b) If $D_1^1 F$ is (2)-differentiable, then F'_{1r} and F'_{2r} are differentiable functions and $[D^2_{1,2}F(t)]^r = [F''_{2r}(t), F''_{1r}(t)]$.
c) If $D_2^1 F$ is (1)-differentiable, then F'_{1r} and F'_{2r} are differentiable functions and $[D^2_{2,1}F(t)]^r = [F''_{2r}(t), F''_{1r}(t)]$.
d) If $D_2^1 F$ is (2)-differentiable, then F'_{1r} and F'_{2r} are differentiable functions and $[D^2_{2,2}F(t)]^r = [F''_{1r}(t), F''_{2r}(t)]$.

For integration of a fuzzy valued function, we will consider the following definition.

Definition 2.5.[38]*Let $F: [a,b] \to \Re_F$. The integral of F on $[a,b]$, denoted by $\int_a^b F(t)\mathrm{d}t$, is defined levelwise by $[\int_a^b F(t)\mathrm{d}t]^r = \int_a^b [F(t)]^r \,\mathrm{d}t, \forall r \in [0,1]$.*

Now, we define some notations which are used for fuzzy fractional calculus throughout this paper:
$C^F[a,b]$ = The space of continuous fuzzy valued functions on $[a,b]$.
$AC^F[a,b]$ = The set of all absolutely continuous fuzzy valued functions.
$L_p^F[a,b] = \{F: [a,b] \to \Re_F; F$ is measurable and
$\int_a^b D(F(x),0)^p \mathrm{d}x < \infty\}, 1 \le p < \infty$.
The generalized H-differentiability was used to expand the definitions of fractional derivatives in the crisp sense for the fuzzy space as follows. For details of fractional derivatives in crisp case, see [44].

Definition 2.6.[45]*Let $0 < \alpha \le 1, F: [a,b] \to \Re_F$ and $F \in C^F[a,b] \cap L^F[a,b]$.*
The fuzzy Riemann-Liouville fractional integral of order α is defined by $(J^\alpha_{a+}F)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{F(t)}{(x-t)^{1-\alpha}}\mathrm{d}t,\ x > a$. It can be written in parametric form as $[(J^\alpha_{a+}F)(x)]^r = [\frac{1}{\Gamma(\alpha)}\int_a^x \frac{F_{1r}(t)}{(x-t)^{1-\alpha}}\mathrm{d}t, \frac{1}{\Gamma(\alpha)}\int_a^x \frac{F_{2r}(t)}{(x-t)^{1-\alpha}}\mathrm{d}t]$.

Definition 2.7.[2] *Let $0 < \alpha \le 1, F: [a,b] \to \Re_F$ and $F \in C^F[a,b] \cap L^F[a,b]$. Then F is said to be Caputo's H-differentiable at x if $(^C D^\alpha_{a+}F)(x) = \dfrac{1}{\Gamma(1-\alpha)}\int_a^x \dfrac{F'(t)}{(x-t)^\alpha}\mathrm{d}t$ exists. We say that F is $^C[(1)-\alpha]-$ differentiable if F is (1)-differentiable, and F is $^C[(2)-\alpha]-$ differentiable if F is (2)-differentiable.*

Now, the extension of the characterization theorems which are introduced for fuzzy differential equations in [4,5] is given.

Theorem 2.5.[6] *Consider the FFDE*

$$(^C D^\alpha_{t_0^+}x)(t) = F(t,x(t)), \qquad x(t_0) = x_0 \in \Re_F \tag{1}$$

where $F: [t_0, t_0 + a] \times \Re_F \to \Re_F$ such that:
(a)$[F(t,x(t))]^r = [F_{1r}(t,x_{1r}(t),x_{2r}(t)), F_{2r}(t,x_{1r}(t),x_{2r}(t))]$
(b) For any $\varepsilon > 0$ there is a $\delta > 0$ such that $|F_{1r}(t,x,z) - F_{1r}(t_1,x_1,z_1)| < \varepsilon$ and $|F_{2r}(t,x,z) - F_{2r}(t_1,x_1,z_1)| < \varepsilon$ for all $r \in [0,1]$, whenever $(t,x,z),(t_1,x_1,z_1) \in [t_0,t_0 + a] \times \Re^2$, $\| (t,x,z) - (t_1,x_1,z_1) \|_{\Re^3} < \delta$ and F_{1r} and F_{2r} are uniformly bounded on any bounded set.
(c) There is an $L > 0$ such that $|F_{1r}(t_2,x_2,z_2) - F_{1r}(t_1,x_1,z_1)| \le L \times max\{|x_2 - x_1|, |z_2 - z_1|\}$ for all $r \in [0,1]$ and $|F_{2r}(t_2,x_2,z_2) - F_{2r}(t_1,x_1,z_1)| \le L \times max\{|x_2 - x_1|, |z_2 - z_1|\}$ for all $r \in [0,1]$.
Then the FFDE (1) is equivalent to the system of ordinary fractional differential equations (OFDEs):

$$\begin{aligned} (^C D^\alpha_{t_0+}x_{1r})(t) &= F_{1r}(t,x_{1r}(t),x_{2r}(t)) \\ (^C D^\alpha_{t_0+}x_{2r})(t) &= F_{2r}(t,x_{1r}(t),x_{2r}(t)) \\ x_{1r}(t_0) = x_{01r}, &\qquad x_{2r}(t_0) = x_{02r} \end{aligned} \tag{2}$$

if x(t) is $^C[(1)-\alpha]$ -differentiable. If x(t) is $^C[(2)-\alpha]$ -differentiable, then (1) is equivalent to the following system of OFDEs:

$$\begin{aligned} (^C D^\alpha_{t_0+}x_{1r})(t) &= F_{2r}(t,x_{1r}(t),x_{2r}(t)) \\ (^C D^\alpha_{t_0+}x_{2r})(t) &= F_{1r}(t,x_{1r}(t),x_{2r}(t)) \\ x_{1r}(t_0) = x_{01r}, &\qquad x_{2r}(t_0) = x_{02r} \end{aligned} \tag{3}$$

Using this theorem, a FFDE can be converted to a system of ODEs of fractional order. Then a numerical method can be applied to solve the resulting system.

3 Second Order Caputo's H-derivative

In this section, we define fuzzy Caputo fractional derivative of order $\beta \in (1,2]$ for a fuzzy function $F: [a,b] \to \Re_F$. Moreover, we give some properties of the mentioned fractional H-derivative.

Definition 3.1.*Let $\beta \in (1,2]$ and $F: [a,b] \to \Re_F$ be such that $F, F' \in C^F[a,b] \cap L^F[a,b]$. Then the second order Caputo's H-derivative of F at $x \in (a,b)$ is defined as*

$$(^C D^\beta_{a+}F)(x) = \frac{1}{\Gamma(2-\beta)}\int_a^x \frac{F''(t)}{(x-t)^{\beta-1}}\mathrm{d}t, x > a. \tag{4}$$

We say that F is $^C[(m,n)-\beta]-$ differentiable for $m,n \in \{1,2\}$ if (4) exists and F is $(m,n)-$differentiable.

Theorem 3.1.*Let $\beta \in (1,2]$ and $F, F' \in AC^F[a,b]$ be such that $[F(x)]^r = [F_{1r}(t), F_{2r}(t)], r \in [0,1]$. Then the second order Caputo's H-derivative exists almost everywhere on (a,b) and*
(i) If F is (1,1)-differentiable, then $[(^C D_{a+}F)(x)]^r$
$= [\frac{1}{\Gamma(2-\beta)}\int_a^x \frac{F''_{1r}(t)}{(x-t)^{(\beta-1)}}\mathrm{d}t, \frac{1}{\Gamma(2-\beta)}\int_a^x \frac{F''_{2r}(t)}{(x-t)^{(\beta-1)}}\mathrm{d}t]$
$= [(^C D^\beta_{a+}F_{1r})(x), (^C D^\beta_{a+}F_{2r})(x)]$.
(ii) If F is (1,2)-differentiable, then $[(^C D_{a+}F)(x)]^r$
$= [\frac{1}{\Gamma(2-\beta)}\int_a^x \frac{F''_{2r}(t)}{(x-t)^{(\beta-1)}}\mathrm{d}t, \frac{1}{\Gamma(2-\beta)}\int_a^x \frac{F''_{1r}(t)}{(x-t)^{(\beta-1)}}\mathrm{d}t]$

$= [({}^C D^{\beta}_{a+} F_{2r})(x), ({}^C D^{\beta}_{a+} F_{1r})(x)]$.

(iii) If F is (2,1)-differentiable, then $[({}^C D_{a+} F)(x)]^r$

$= [\frac{1}{\Gamma(2-\beta)} \int_a^x \frac{F''_{2r}(t)}{(x-t)^{(\beta-1)}} \mathrm{d}t, \frac{1}{\Gamma(2-\beta)} \int_a^x \frac{F''_{1r}(t)}{(x-t)^{(\beta-1)}} \mathrm{d}t]$

$= [({}^C D^{\beta}_{a+} F_{2r})(x), ({}^C D^{\beta}_{a+} F_{1r})(x)]$.

(iv) If F is (2,2)-differentiable, then $[({}^C D_{a+} F)(x)]^r$

$= [\frac{1}{\Gamma(2-\beta)} \int_a^x \frac{F''_{1r}(t)}{(x-t)^{(\beta-1)}} \mathrm{d}t, \frac{1}{\Gamma(2-\beta)} \int_a^x \frac{F''_{2r}(t)}{(x-t)^{(\beta-1)}} \mathrm{d}t]$

$= [({}^C D^{\beta}_{a+} F_{1r})(x), ({}^C D^{\beta}_{a+} F_{2r})(x)]$.

*Proof.*Using Theorem 2.4, the proof results directly.

Theorem 3.2.*Let* $\beta \in (1,2]$ *and* $F, F' \in AC^F[a,b]$.

(1) If F is (1,1)-differentiable, then
$(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x) = F(x) \ominus F(a) \ominus F'(a)(x-a)$.

(2) If F is (1,2)-differentiable, then
$(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x) = -F(a) + (-F'(a))(x-a) \ominus (-F(x))$.

(3) If F is (2,1)-differentiable, then
$(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x) = -F(a) \ominus F'(a)(x-a) \ominus (-F(x))$.

(4) If F is (2,2)-differentiable, then
$(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x) = F(x) \ominus F(a) + (-F'(a))(x-a)$.

*Proof.*Let $[F(x)]^r = [F_{1r}(t), F_{2r}(t)]$ for $r \in [0,1]$. Then we have for the real valued functions F_{1r} and F_{2r}, $(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F_{1r})(x) = F_{1r}(x) - F_{1r}(a) - F'_{1r}(a)(x-a)$ and $(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F_{2r})(x) = F_{2r}(x) - F_{2r}(a) - F'_{2r}(a)(x-a)$.
Assume that F is (1,1)-differentiable or (2,2)-differentiable, then by Theorem (3.1), we can write $[({}^C D^{\beta}_{a+} F)(x)]^r = [({}^C D^{\beta}_{a+} F_{1r})(x), ({}^C D^{\beta}_{a+} F_{2r})(x)]$. Hence $[(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x)]^r = [(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F_{1r})(x), (J^{\beta}_{a+} {}^C D^{\beta}_{a+} F_{2r})(x)]$ $= [F_{1r}(x) - F_{1r}(a) - F'_{1r}(a)(x-a), F_{2r}(x) - F_{2r}(a) - F'_{2r}(a)(x-a)]$.
So $(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x) = F(x) \ominus F(a) \ominus F'(a)(x-a)$ if F is (1,1)-differentiable, and $(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x) = F(x) \ominus F(a) + (-F'(a))(x-a)$ if F is (2,2)-differentiable.
Now, if F is (1,2)-differentiable or (2,1)-differentiable, then from Theorem (3.1) we have $[({}^C D^{\beta}_{a+} F)(x)]^r = [({}^C D^{\beta}_{a+} F_{2r})(x), ({}^C D^{\beta}_{a+} F_{1r})(x)]$. So $[(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x)]^r = [(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F_{2r})(x), (J^{\beta}_{a+} {}^C D^{\beta}_{a+} F_{1r})(x)]$ $= [F_{2r}(x) - F_{2r}(a) - F'_{2r}(a)(x-a), F_{1r}(x) - F_{1r}(a) - F'_{1r}(a)(x-a)]$. Hence, $(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x) = -F(a) + (-F'(a))(x-a) \ominus (-F(x))$ if F is (1,2)-differentiable,
and $(J^{\beta}_{a+} {}^C D^{\beta}_{a+} F)(x) = -F(a) \ominus F'(a)(x-a) \ominus (-F(x))$ if F is (2,1)-differentiable.

4 Second Order Fuzzy Fractional Differential Equations

In this section, we study FFDEs of the form

$$\begin{aligned} &({}^C D^{\beta}_{a^+} x)(t) = h(t)x'(t) + F(t, x(t)), 1 < \beta \leq 2, t \geq a \\ &x(a) = \alpha, \qquad x'(a) = \alpha' \end{aligned} \tag{5}$$

where $h(t)$ is a continuous real valued function with nonnegative values on $[a,b], F\colon [a,b] \times \Re_F \to \Re_F$ is a linear or nonlinear continuous fuzzy function, and $\alpha, \alpha' \in \Re_F$. An (m,n)-solution of (5) is an ${}^C[(m,n)-\beta]-$differentiable function $x\colon [a,b] \to \Re_F$ that satisfies (5). To solve this problem, we convert it to a system of second order fractional differential equations based on the selection of the derivative type. This system will be called (m,n)-system.
Let $[F(t,x(t))]^r = [F_{1r}(t, x_{1r}(t), x_{2r}(t)), F_{2r}(t, x_{1r}(t), x_{2r}(t))]$, $[x(t)]^r = [x_{1r}(t), x_{2r}(t)], [x(a)]^r = [x_{1r}(a), x_{2r}(a)] = [\alpha_{1r}, \alpha_{2r}]$ and $[x'(a)]^r = [x'_{1r}(a), x'_{2r}(a)] = [\alpha'_{1r}, \alpha'_{2r}]$ be the r-cut representations of $F(t,x(t))$ and $x(t)$. Then (5) can be translated to one of the following systems:

(1,1)-system:

$$\begin{cases} ({}^C D^{\beta}_{a^+} x_{1r})(t) = h(t)x'_{1r}(t) + F_{1r}(t, x_{1r}(t), x_{2r}(t)), \\ ({}^C D^{\beta}_{a^+} x_{2r})(t) = h(t)x'_{2r}(t) + F_{2r}(t, x_{1r}(t), x_{2r}(t)), \\ x_{1r}(a) = \alpha_{1r}, \qquad x'_{1r}(a) = \alpha'_{1r}, \\ x_{2r}(a) = \alpha_{2r}, \qquad x'_{2r}(a) = \alpha'_{2r} \end{cases} \tag{6}$$

(1,2)-system:

$$\begin{cases} ({}^C D^{\beta}_{a^+} x_{2r})(t) = h(t)x'_{1r}(t) + F_{1r}(t, x_{1r}(t), x_{2r}(t)), \\ ({}^C D^{\beta}_{a^+} x_{1r})(t) = h(t)x'_{2r}(t) + F_{2r}(t, x_{1r}(t), x_{2r}(t)), \\ x_{1r}(a) = \alpha_{1r}, \qquad x'_{1r}(a) = \alpha'_{1r}, \\ x_{2r}(a) = \alpha_{2r}, \qquad x'_{2r}(a) = \alpha'_{2r} \end{cases} \tag{7}$$

(2,1)-system:

$$\begin{cases} ({}^C D^{\beta}_{a^+} x_{2r})(t) = h(t)x'_{2r}(t) + F_{1r}(t, x_{1r}(t), x_{2r}(t)), \\ ({}^C D^{\beta}_{a^+} x_{1r})(t) = h(t)x'_{1r}(t) + F_{2r}(t, x_{1r}(t), x_{2r}(t)), \\ x_{1r}(a) = \alpha_{1r}, \qquad x'_{1r}(a) = \alpha'_{2r}, \\ x_{2r}(a) = \alpha_{2r}, \qquad x'_{2r}(a) = \alpha'_{1r} \end{cases} \tag{8}$$

(2,2)-system:

$$\begin{cases} ({}^C D^{\beta}_{a^+} x_{1r})(t) = h(t)x'_{2r}(t) + F_{1r}(t, x_{1r}(t), x_{2r}(t)), \\ ({}^C D^{\beta}_{a^+} x_{2r})(t) = h(t)x'_{1r}(t) + F_{2r}(t, x_{1r}(t), x_{2r}(t)), \\ x_{1r}(a) = \alpha_{1r}, \qquad x'_{1r}(a) = \alpha'_{2r}, \\ x_{2r}(a) = \alpha_{2r}, \qquad x'_{2r}(a) = \alpha'_{1r} \end{cases} \tag{9}$$

Theorem 4.1.*Let $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$ be an (m,n)-solution of (5). Then $x_{1r}(t)$ and $x_{2r}(t)$ solve the corresponding (m,n)-system for $n, m \in \{1,2\}$. Moreover, if $x_{1r}(t)$ and $x_{2r}(t)$ solve the (m,n)-system for each $r \in [0,1]$, $[x_{1r}(t), x_{2r}(t)]$ has valid level sets, and $x(t)$ is ${}^C[(m,n) - \beta]$−differentiable, then $x(t)$ is an (m,n)-solution of (5).*

*Proof.*The same as the proofs of theorems (4.2) and (4.3) in [3]

Algorithm 4.1 To find solutions of (5), we follow the steps:
Step1: Assume that $x(t)$ is ${}^C[(m,n) - \beta]$− differentiable and convert (5) to the corresponding (m,n)-system.
Step2: Solve the system.
Step3: Ensure that the resulting solution satisfies Theorems (2.3)and (3.1)

5 The reproducing kernel Hilbert space method for Solving FFIVPs

To obtain (m,n)-solution of (5), we apply the RKHS method to solve the corresponding (m,n)-system. We give a summary of the procedure to obtain the analytic and approximate (1,1)-solutions which is equivalent to the solution of (6). In fact, the same technique can be employed to construct other types of solutions. For the details of this method, see [30,46,47].

Algorithm 5.1
(1)Use the transform $y_{1r}(t) = x_{1r}(t) - \alpha_{1r} - (t-a)\alpha'_{1r}$, $y_{2r}(t) = x_{2r}(t) - \alpha_{2r} - (t-a)\alpha'_{2r}$ to homogenize the initial conditions and rewrite (6) in the form:

$$\begin{aligned} ({}^C D^{\beta}_{a^+} y_{1r})(t) &= h_{1r}(t, y_{1r}(t), y_{2r}(t), y'_{1r}(t), y'_{2r}(t)),\\ ({}^C D^{\beta}_{a^+} y_{2r})(t) &= h_{2r}(t, y_{1r}(t), y_{2r}(t), y'_{1r}(t), y'_{2r}(t)),\\ y_{1r}(a) = y'_{1r}(a) &= y_{2r}(a) = y'_{2r}(a) = 0 \end{aligned} \tag{10}$$

(2)Apply the operator J^{β}_{a+} to the both sides of the two differential equations in (10) to get
$y_{jr}(t) = H_{jr}(t, y_{1r}(t), y_{2r}(t), y'_{1r}(t), y'_{2r}(t))$
$= \frac{1}{\Gamma(\beta)} \int_a^t \frac{h_{jr}(s, y_{1r}(s), y_{2r}(s), y'_{1r}(s), y'_{2r}(s))}{(t-s)^{1-\beta}} dt, t > a, j = 1,2.$
(3) Construct reproducing kernel functions of certain spaces:
i.$W_2^1[a,b] = \{u\colon [a,b] \to \Re\colon u \in AC[a,b], u' \in L_2[a,b]\}$ with inner product for $u, v \in W_2^1[a,b]$ given by $\langle u,v\rangle_{W_2^1} = \int_a^b (u(t)v(t) + u'(t)v'(t))dt$ and norm: $\|u\|_{W_2^1} = \sqrt{\langle u(t), u(t)\rangle_{W_2^1}}$. Its reproducing function has the form $R_t(s) = \frac{1}{2sinh(b-a)}[cosh(t+s-b-a) + cosh(|t-s| - b + a)]$.

ii. $W_2^3[a,b] = \{u\colon u, u', u'' \in AC[a,b], u''' \in L_2[a,b],$ $u(a) = u'(a) = 0\}$ with inner product for $u, v \in W_2^3[a,b]$ given by $\langle u,v\rangle_{W_2^3} = u''(a)v''(a) + \int_a^b u'''(t)v'''(t))dt$ and norm: $\|u\|_{W_2^3} = \sqrt{\langle u(t), u(t)\rangle_{W_2^3}}$. The reproducing function of $W_2^3[a,b]$ is $G_t(s) = \begin{cases} g(t,s) & s \le t \\ g(s,t) & s > t \end{cases}$ where $g(t,s) = -\frac{1}{120}(a-s)^2(6a^3 + 5ts^2 - s^3 - 10t^2(3+s) - 3a^2(10 + 5t + s) + 2a(5t^2 - s^2 + 5t(6+s)))$.

iii. $N^m[a,b] = W_2^m[a,b] \bigoplus W_2^m[a,b] = \{(u_1(t), u_2(t))^T\colon u_1, u_2 \in W_2^m[a,b]\}, m = 1,2$ The inner product and the norm of $u(t) = (u_1(t), u_2(t))^T$ and $v(t) = (v_1(t), v_2(t))^T$ in $N^m[a,b]$ are given by $\langle u,v\rangle_{N^m} = \sum_{i=1}^2 \langle u_i(t), v_i(t)\rangle_{W_2^m}$ and $\|u\|_{N^m} = \sqrt{\sum_{i=1}^2 \|u_i\|^2_{W_2^m}}$, respectively.
(4) Define the operator $I_{jr}\colon W_2^3[a,b] \to W_2^1[a,b]$ by $I_{jr} y_{jr}(t) = y_{jr}(t), j = 1,2,$ and let $I_r = diag(I_{1r}, I_{2r})$. Obviously, $I_{jr}, j = 1,2$ are linear and bounded. Consequently, I_r is also a bounded linear operator such that $I_r : N^3[a,b] \to N^1[a,b]$. Put $G_r = (G_{1r}, G_{2r})^T$ and $y_r = (y_{1r}, y_{2r})^T$ to rewrite (10) in the form $I_r y_r(t) = G_r(t, y_r(t), y'_r(t)), y'(a) = y(a) = 0$.
(5) Consider the countable dense set $\{t_i\}_{i=1}^{\infty}$, and let $\varphi_{ij}(t) = G_{t_i}(t)e_j$ and $\Psi_{ij}(t) = I^*_r \varphi_{ij}(t), j = 1,2$ to construct an orthogonal function system $\{\Psi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ of the space $N^3[a,b]$. Then use the Gram-Schmidt orthogonalization process on it to form the orthonormal function system $\{\overline{\Psi_{ij}}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ of $N^3[a,b]$.
(6) Using this operator, the approximate (1,1)-solution of (10) has the form:

$$y^n_r(t) = \sum_{i=1}^{n} \sum_{j=1}^{2} \sum_{l=1}^{i} \sum_{k=1}^{j} \beta_{kil} G_{kr}(t_l, y_r(t_l), y'_r(t_l)) \overline{\Psi_{ij}}(t), \tag{11}$$

which converges to the analytic solution:

$$y_r(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{2} \sum_{l=1}^{i} \sum_{k=1}^{j} \beta_{kil} G_{kr}(t_l, y_r(t_l), y'_r(t_l)) \overline{\Psi_{ij}}(t),$$

where β_{kil} are the orthogonalization coefficients. So the approximate solution $x_r(t)$ of (5) is $x^n_r(t) = y^n_r(t) + \alpha_r + (t-a)\alpha'_r$.

Numerical Examples

In this subsection, we give examples of second order FFIVPs and solve them using the RKHSM. Our

computations are performed using Mathematica7.0.
Example1 Consider the following FFIVP:

$$({}^{C}D^{\beta}_{0^+}x)(t) = \sigma, \qquad 1 < \beta \leq 2, t \in [0,1]$$

$$x(0) = \gamma, x'(0) = \alpha,$$

where $\sigma = \alpha = \gamma$ are the fuzzy numbers whose r-cut representation is $[r-1, 1-r]$.
Depending on the type of differentiability, we have the following systems:

$$\textbf{(1,1)-system:} \begin{cases} ({}^{C}D^{\beta}_{0^+}x_{1r})(t) = r-1, \\ ({}^{C}D^{\beta}_{0^+}x_{2r})(t) = 1-r, \\ x_{1r}(0) = x'_{1r}(0) = r-1, \\ x_{2r}(0) = x'_{2r}(0) = 1-r. \end{cases}$$

$$\textbf{(1,2)-system:} \begin{cases} ({}^{C}D^{\beta}_{0^+}x_{1r})(t) = 1-r, \\ ({}^{C}D^{\beta}_{0^+}x_{2r})(t) = r-1, \\ x_{1r}(0) = x'_{1r}(0) = r-1, \\ x_{2r}(0) = x'_{2r}(0) = 1-r. \end{cases}$$

$$\textbf{(2,1)-system:} \begin{cases} ({}^{C}D^{\beta}_{0^+}x_{1r})(t) = 1-r, \\ ({}^{C}D^{\beta}_{0^+}x_{2r})(t) = r-1, \\ x_{1r}(0) = x'_{2r}(0) = r-1, \\ x_{2r}(0) = x'_{1r}(0) = 1-r. \end{cases}$$

$$\textbf{(2,2)-system:} \begin{cases} ({}^{C}D^{\beta}_{0^+}x_{1r})(t) = r-1, \\ ({}^{C}D^{\beta}_{0^+}x_{2r})(t) = 1-r, \\ x_{1r}(0) = x'_{2r}(0) = r-1, \\ x_{2r}(0) = x'_{1r}(0) = 1-r. \end{cases}$$

Applying Theorem (3.2), the exact solutions are:
(1,1)-solution:
$[x(t)]^r = [r-1, 1-r](\frac{t^{\beta}}{\Gamma(\beta+1)} + t + 1), t \in [0,1].$
(1,2)-solution:
$[x(t)]^r = [r-1, 1-r]\frac{-t^{\beta}}{\Gamma(\beta)} + t + 1), t \in [0,1].$
(2,1)-solution:
$[x(t)]^r = [r-1, 1-r]\frac{-t^{\beta}}{\Gamma(\beta+1)} - t + 1), t \in (0, \sqrt{3}-1).$
(2,2)-solution:
$[x(t)]^r = [r-1, 1-r]\frac{t^{\beta}}{\Gamma(\beta+1)} - t + 1), t \in [0,1].$
Using the RKHS method with $n = 100$ and $m = 5$, some numerical results are given in Table1 and Figures 1 and 2.
Example 2 Consider the following FFIVP:

$$({}^{C}D^{\beta}_{0^+}x)(t) + x(t) = \sigma, 1 < \beta \leq 2, t \in [0,1]$$

Table 1: The error of example1 at different values of t and r when $\beta = 1.9$.

r	0.25	0.5	0.75
t	(1,1)-solution		
0.1	8.23357×10^{-5}	5.48905×10^{-5}	2.74452×10^{-5}
0.2	1.54339×10^{-4}	1.02893×10^{-4}	5.14464×10^{-5}
0.3	2.22572×10^{-4}	1.48381×10^{-4}	7.41906×10^{-5}
0.4	2.88491×10^{-4}	1.92327×10^{-4}	9.61636×10^{-5}
0.5	3.52749×10^{-4}	2.35166×10^{-4}	1.17583×10^{-4}
0.6	4.15716×10^{-4}	2.77144×10^{-4}	1.38572×10^{-4}
0.7	4.77632×10^{-4}	3.18422×10^{-4}	1.59211×10^{-4}
0.8	5.38664×10^{-4}	3.59109×10^{-4}	1.79555×10^{-4}
0.9	5.98932×10^{-4}	3.99288×10^{-4}	1.99644×10^{-4}
1	6.58535×10^{-4}	4.39023×10^{-4}	2.19512×10^{-4}
t	(1,2)-solution		
0.1	8.2336×10^{-5}	5.48905×10^{-5}	2.74452×10^{-5}
0.2	1.54339×10^{-4}	1.02893×10^{-4}	5.14464×10^{-5}
0.3	2.22572×10^{-4}	1.48381×10^{-4}	7.41906×10^{-5}
0.4	2.88491×10^{-4}	1.92327×10^{-4}	9.61636×10^{-5}
0.5	3.52749×10^{-4}	2.35166×10^{-4}	1.17583×10^{-4}
0.6	4.15716×10^{-4}	2.77144×10^{-4}	1.38572×10^{-4}
0.7	4.77632×10^{-4}	3.18422×10^{-4}	1.59211×10^{-4}
0.8	5.38664×10^{-4}	3.59109×10^{-4}	1.79555×10^{-4}
0.9	5.98932×10^{-4}	3.99288×10^{-4}	1.99644×10^{-4}
1	6.58535×10^{-4}	4.39023×10^{-4}	2.19512×10^{-4}
t	(2,1)-solution		
0.1	8.23357×10^{-5}	5.48905×10^{-5}	2.74452×10^{-5}
0.2	1.54339×10^{-4}	1.02893×10^{-4}	5.14464×10^{-5}
0.3	2.22572×10^{-4}	1.48381×10^{-4}	7.41906×10^{-5}
0.4	2.88491×10^{-4}	1.92327×10^{-4}	9.61636×10^{-5}
0.5	3.52749×10^{-4}	2.35166×10^{-4}	1.17583×10^{-4}
0.6	4.15716×10^{-4}	2.77144×10^{-4}	1.38572×10^{-4}
0.7	4.77632×10^{-4}	3.18422×10^{-4}	1.59211×10^{-4}
t	(2,2)-solution		
0.1	8.23357×10^{-5}	5.48905×10^{-5}	2.744524×10^{-5}
0.2	1.54339×10^{-4}	1.02893×10^{-4}	5.14464×10^{-5}
0.3	2.22572×10^{-4}	1.48381×10^{-4}	7.41906×10^{-5}
0.4	2.88491×10^{-4}	1.92327×10^{-4}	9.61636×10^{-5}
0.5	3.52749×10^{-4}	2.35166×10^{-4}	1.17583×10^{-4}
0.6	4.15716×10^{-4}	2.77144×10^{-4}	1.38572×10^{-4}
0.7	4.77632×10^{-4}	3.18422×10^{-4}	1.59211×10^{-4}
0.8	5.38664×10^{-4}	3.59109×10^{-4}	1.79555×10^{-4}
0.9	5.98932×10^{-4}	3.99288×10^{-4}	1.99644×10^{-4}
1	6.58535×10^{-4}	4.39023×10^{-4}	2.19512×10^{-4}

subject to $x(0) = \lambda, x'(0) = \alpha$, where σ is the fuzzy number with r-cut representation $[r, 2-r]$ and $[\alpha]^r = [\lambda]^r = [r-1, 1-r]$. Depending on the type of differentiability, we have the following systems:

$$\textbf{(1,1)-system:} \begin{cases} ({}^{C}D^{\beta}_{0^+}x_{1r})(t) + x_{1r}(t) = r, \\ ({}^{C}D^{\beta}_{0^+}x_{2r})(t) + x_{2r}(t) = 2-r, \\ x_{1r}(0) = x'_{1r}(0) = r-1, \\ x_{2r}(0) = x'_{2r}(0) = 1-r. \end{cases}$$

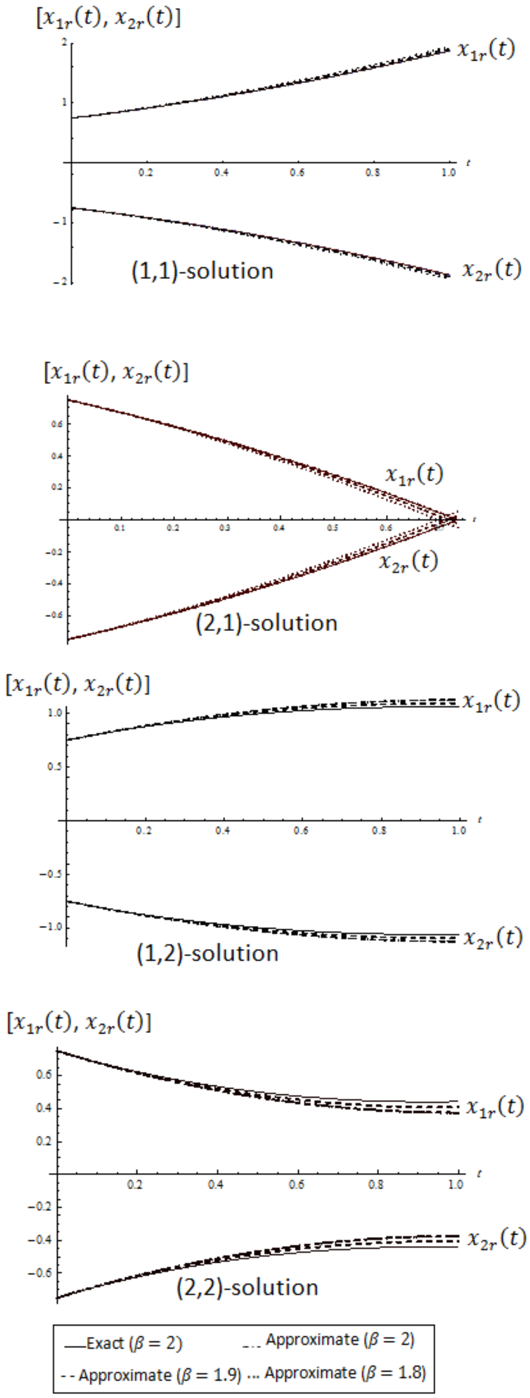

Fig. 1: Exact and approximate solutions $x(t)$ for different values of β at $r = 0.25$ for example 1.

The exact solution of this system for $\beta = 2$ is $[x(t)]^r = [r, 2-r](1+sint) - sint - cost$ which is not (1,1)-differentiable. Hence, in this case, no solution for the FFIVP exists.

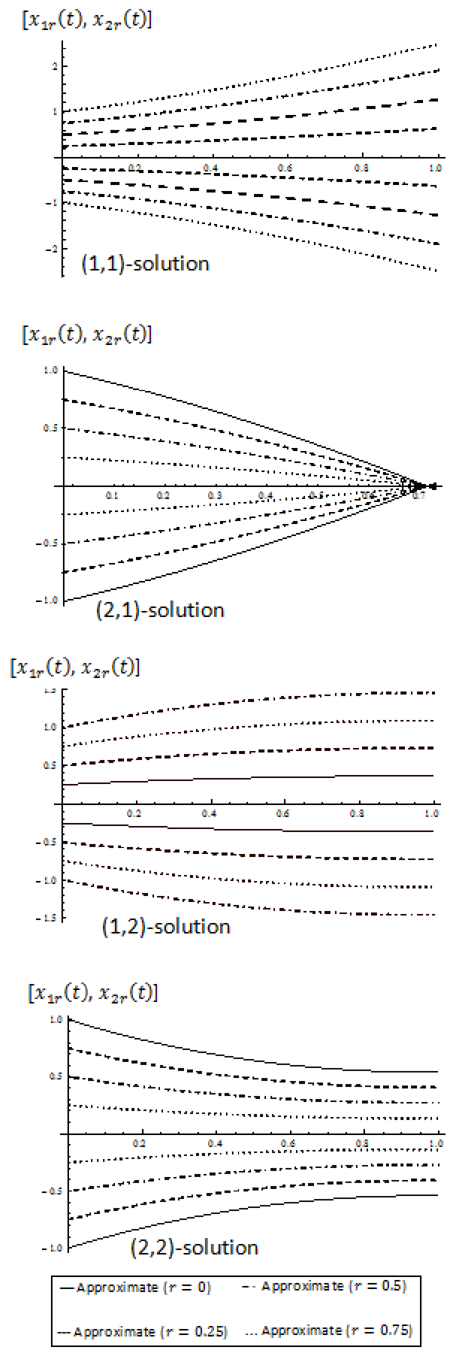

Fig. 2: Approximate solutions for different values of r at $\beta = 1.9$ for example 1.

$$\textbf{(1,2)-system:} \begin{cases} ({}^C D^{\beta}_{0+} x_{2r})(t) + x_{1r}(t) = r, \\ ({}^C D^{\beta}_{0+} x_{1r})(t) + x_{2r}(t) = 2 - r, \\ x_{1r}(0) = x'_{1r}(0) = r - 1, \\ x_{2r}(0) = x'_{2r}(0) = 1 - r. \end{cases}$$

The exact solution of this system when $\beta = 2$ is $[x(t)]^r = [r, 2-r](1+sinht) - sinht - cost$ which is not (1,2)-differentiable.

$$\textbf{(1,2)-system:} \begin{cases} ({}^C D^{\beta}_{0+} x_{2r})(t) + x_{1r}(t) = r, \\ ({}^C D^{\beta}_{0+} x_{1r})(t) + x_{2r}(t) = 2-r, \\ x_{1r}(0) = x'_{2r}(0) = r-1, \\ x_{2r}(0) = x'_{1r}(0) = 1-r. \end{cases}$$

The exact solution of this system when $\beta = 2$ is $[x(t)]^r = [r, 2-r](1-sinht) + sinht - cost$ which is (2,1)-differentiable for $t \in (0, ln(1+\sqrt{2}))$.

Using the RKHS method with n=100 and m=5, some numerical results are given in Table2 and Figure3.

Table 2: The fuzzy approximate (2,1)-solution $[x_{1r}(0.4), x_{2r}(0.4)$ of example 2 at different values of β and r.

r	$\beta = 2$	Error ($\beta = 2$)
0	[-0.510308,0.668146]	6.44195789×10^{-7}
0.25	[-0.363001,0.52084]	4.41377867×10^{-6}
0.5	[-0.215694,0.373533]	9.47175312×10^{-6}
0.75	[-0.0683874,0.2262262]	1.45297276×10^{-5}
1	[0.078919,0.078919]	1.95877020×10^{-5}
r	$\beta = 1.9$	$\beta = 1.8$
0	[-0.493755,0.678996]	[-0.484393,0.685333]
0.25	[-0.347161,0.532403]	[-0.338178,0.539118]
0.5	[-0.200567, 0.385808]	[-0.191961,0.392902]
0.75	[-0.053975,0.239215]	[-0.045745,0.246686]
1	[0.092620,0.092620]	[0.100470,0.100470]

$$\textbf{(2,2)-system:} \begin{cases} ({}^C D^{\beta}_{0+} x_{1r})(t) + x_{1r}(t) = r, \\ ({}^C D^{\beta}_{0+} x_{2r})(t) + x_{2r}(t) = 2-r, \\ x_{1r}(0) = x'_{2r}(0) = r-1, \\ x_{2r}(0) = x'_{1r}(0) = 1-r. \end{cases}$$

The exact solution for $\beta = 2$ is $[x(t)]^r = [r, 2-r](1-sint) + sint - cost$ which is (2,2)-differentiable for $t \in (0, \frac{\pi}{2})$.

Using the RKHS method with n=100 and m=5, some numerical results are given in Table3 and Figure4.

Example 3 Consider the following FFIVP: $({}^C D^{\beta}_{0+} x)(t) = x'(t) + t + 1, \quad 1 < \beta \le 2, t \in [0,1]$, $x(0) = \lambda, x'(0) = \alpha$, where $[\lambda]^r = [\alpha]^r = [r-2, 1-2r]$. Depending on the type of differentiability, we have the following systems:

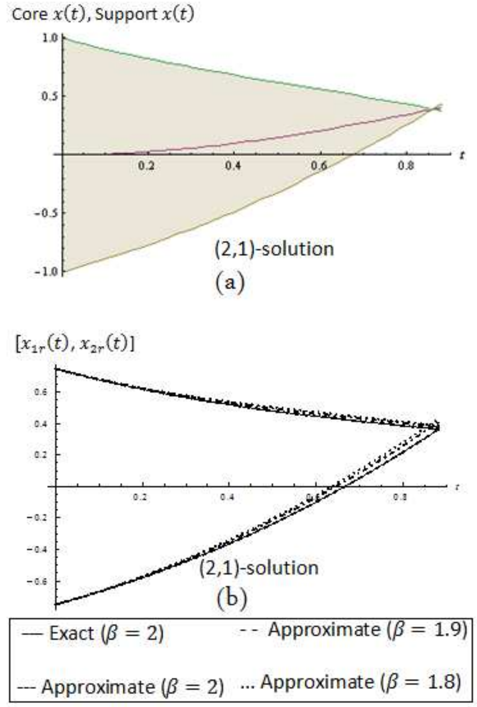

Fig. 3: a)The core and the support of the fuzzy (2,1)- approximate solutions at $\beta = 1.8$, b) Approximate (2,1)-solutions for different values of β at $r = 0.25$ for example 2.

Table 3: The fuzzy approximate (2,2)-solution $[x_{1r}(0.5), x_{2r}(0.5)$ of example 2 at different values of β and r.

r	$\beta = 2$	Error ($\beta = 2$)
0	[-0.398179,0.642991]	4.34399357×10^{-7}
0.25	[-0.268033,0.512845]	3.12570815×10^{-6}
0.5	[-0.137886,0.382699]	5.81701694×10^{-6}
0.75	[-0.007740,0.252553]	8.50832573×10^{-6}
1	[0.122406,0.122406]	1.11996345×10^{-5}
r	$\beta = 1.9$	$\beta = 1.8$
0	[-0.384033,0.666088]	[-0.376094,0.678837]
0.25	[-0.252768,0.534823]	[-0.244228,0.546971]
0.5	[-0.121503,0.403557]	[-0.112362,0.415104]
0.75	[0.009762,0.272292]	[0.019505,0.283238]
1	[0.141027,0.141027]	[0.151371,0.151371]

$$\textbf{(1,1)-system:} \begin{cases} ({}^C D^{\beta}_{0+} x_{1r})(t) = x'_{1r}(t) + t + 1, \\ ({}^C D^{\beta}_{0+} x_{2r})(t) = x'_{2r}(t) + t + 1, \\ x_{1r}(0) = x'_{1r}(0) = r-2, \\ x_{2r}(0) = x'_{2r}(0) = 1-2r. \end{cases}$$

The exact solution of this system for $\beta = 2$ is $x_{1r}(t) = re^t - \frac{t^2}{2} - 2t - 2, x_{2r}(t) = (3-2r)e^t - \frac{t^2}{2} - 2t - 2$

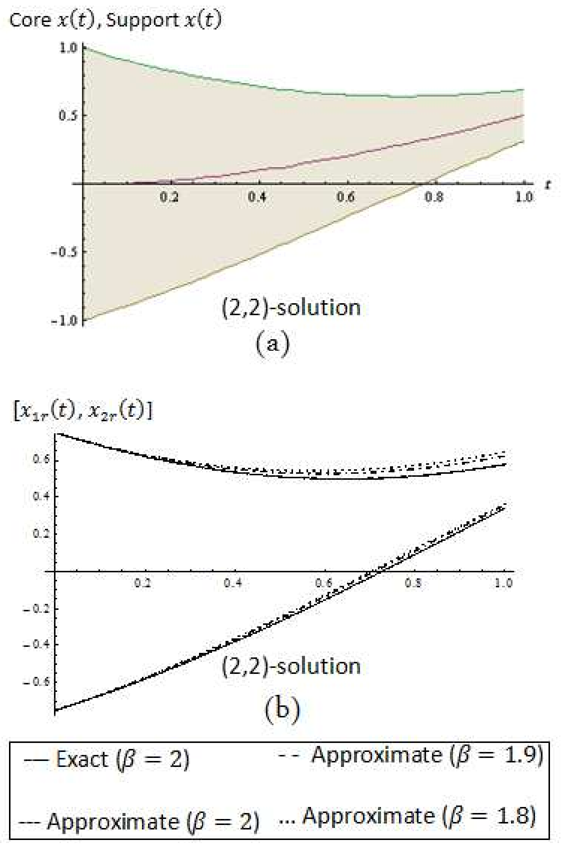

Fig. 4: a)The core and the support of the fuzzy (2,2)- approximate solutions at $\beta = 1.8$, b) Approximate (2,2)-solutions for different values of β at $r = 0.25$ for example 2.

$$\textbf{(1,2)-system:} \begin{cases} ({}^{C}D^{\beta}_{0+}x_{2r})(t) = x'_{1r}(t) + t + 1, \\ ({}^{C}D^{\beta}_{0+}x_{1r})(t) = x'_{2r}(t) + t + 1, \\ x_{1r}(0) = x'_{1r}(0) = r - 2, \\ x_{2r}(0) = x'_{2r}(0) = 1 - 2r. \end{cases}$$

The exact solution of this system for $\beta = 2$ is
$x_{1r}(t) = re^t + 3(1-r)cosht - \frac{t^2}{2} - 2t - 5 + 3r,$
$x_{2r}(t) = (3-2r)e^t + 3(r-1)cosht - \frac{t^2}{2} - 2t + 1 - 3r$
which is not (1,2)-differentiable.

$$\textbf{(2,1)-system:} \begin{cases} ({}^{C}D^{\beta}_{0+}x_{2r})(t) = x'_{2r}(t) + t + 1, \\ ({}^{C}D^{\beta}_{0+}x_{1r})(t) = x'_{1r}(t) + t + 1, \\ x_{1r}(0) = x'_{2r}(0) = r - 2, \\ x_{2r}(0) = x'_{1r}(0) = 1 - 2r. \end{cases}$$

The exact solution of this system for $\beta = 2$ is
$x_{1r}(t) = (3-2r)e^t - \frac{t^2}{2} - 2t - 5 + 3r,$
$x_{2r}(t) = re^t - \frac{t^2}{2} - 2t + 1 - 3r$
which is (2,1)-differentiable for $t \in (0, ln2)$.

$$\textbf{(2,2)-system:} \begin{cases} ({}^{C}D^{\beta}_{0+}x_{1r})(t) = x'_{2r}(t) + t + 1, \\ ({}^{C}D^{\beta}_{0+}x_{2r})(t) = x'_{1r}(t) + t + 1, \\ x_{1r}(0) = x'_{2r}(0) = r - 2, \\ x_{2r}(0) = x'_{1r}(0) = 1 - 2r. \end{cases}$$

The exact solution of this system for $\beta = 2$ is $[x(t)]^r = \frac{1}{2}e^{-t}((3r-3, 3-3r) - \frac{1}{2}e^{-t}(4e^t - 3e^{2t} + e^{2t}r + 4e^t t + e^t t^2)$. Using the RKHS method with n=100 and m=5, the numerical results are given in Figure5 and Table4.

Table 4: The fuzzy approximate solutions of example 3 at different values of β and r.

$r:$	0.25	0.5
Approximate (1,1)-solution $[x_{1r}(0.5), x_{2r}(0.5)]$		
$\beta = 2$	[-2.712788,0.996826]	[-2.300609,0.172467]
Error	3.123591×10^{-5}	2.474548×10^{-5}
$\beta = 1.9$	[-2.730153,1.051163]	[-2.310007,0.210871]
$\beta = 1.8$	[-2.753118,-2.753118]	[-2.322627,0.260319]
Approximate (2,1)-solution $[x_{1r}(0.6), x_{2r}(0.6)]$		
$\beta = 2$	[-1.074675,-0.674431]	[-1.235732,-0.968902]
Error	2.773201×10^{-5}	3.034650×10^{-5}
$\beta = 1.9$	[-0.996812,-0.697307]	[-1.180255,-0.980585]
$\beta = 1.8$	[-0.896953,-0.727305]	[-1.109253,-0.996155]
Approximate (2,2)-solution $[x_{1r}(0.5), x_{2r}(0.5)]$		
$\beta = 2$	[-1.540386,-0.175658]	[-1.519009,-0.609190]
Error	3.568930×10^{-6}	1.050045×10^{-5}
$\beta = 1.9$	[-1.537332,-0.143039]	[-1.514875,-0.585346]
$\beta = 1.8$	[-1.529377,-0.104349]	[-1.506915,-0.556897]

6 Conclusions

In this paper, we present a definition of second order Caputo's H-derivative and its r-cut representations under different types of differentiability. We give the fuzzy forms of the Riemann-Liouville fractional integral when applied to the Caputo's H-derivative of order $\beta \in (1,2]$ of a fuzzy function. The generalized characterization theorem allows us to translate the FFDE into four systems of fractional differential equations and solve them instead of solving the FFDE. For a numerical solution, , we apply a modified RKHSM to obtain analytic and approximate solutions in series form in term of their parametric forms in the space $W^3[a,b] \oplus W^3[a,b]$ .Several examples are given to show the effectiveness of the proposed method. To see the effects of the fractional derivative on the solution, we solve the same FDEs for different values of the fractional order. The results shows that the solutions

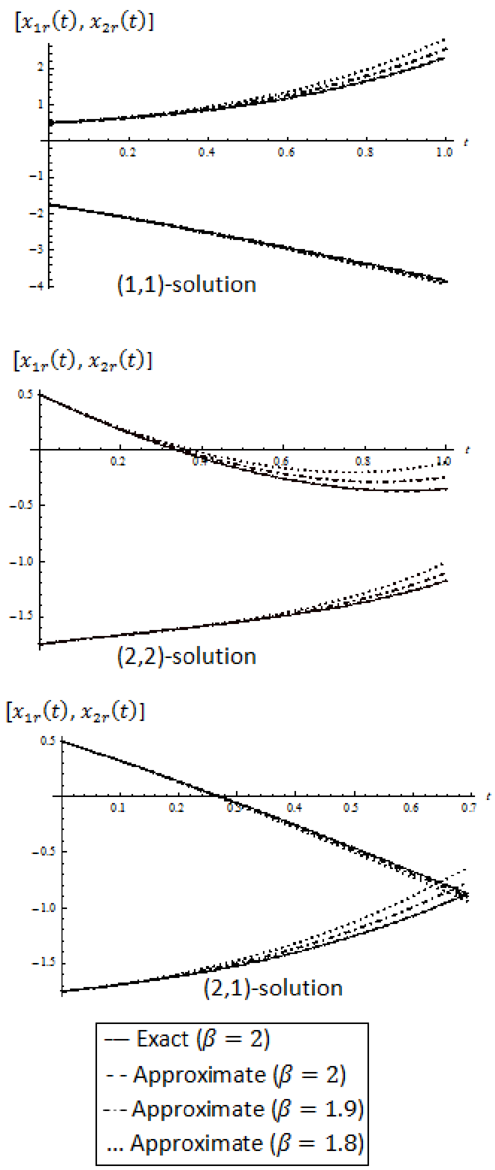

Fig. 5: The approximate solutions for different values of β at r=0.25, for example 3.

of FFDEs approach the solution of FDEs as the fractional order approaches the integer order.

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Shatha Hasan graduated from the university of Jordan in 2006 with a B.S. in Mathematics; and an M.S. degree in Mathematics in 2009. In 2016, she received her Ph.D. from the University of Jordan in applied mathematics. Her research interests are in the areas of applied mathematics specially in solving fuzzy differential equations of fractional order.

Ahmad Alawneh received his Ph.D from Pennsylvania state university(USA)in 1975. He then began work at the university of Jordan in 1975 as an assistant professor of mathematics and promoted to full professor in 1997. His research interests focus on analytical solutions of boudary value problems in different fields of mathematical physics and engineering as well as on analytical and numerical solutions of fractional differential equations.

Mohammad Al-Momani graduated from Al-Balqa Applied University in 2006 with a B.S. in Computer Information Systems (CIS) and in M.S. degree in CIS in 2009 from Near East University (Cyprus). In 2014, he received his Ph.D in Management Information Systems (MIS) from Girne American University (Cyprus). Dr. Al-Momani is now an Assistant Professor at Zarqa University (Jordan) and is currently serving as a head of MIS department. His research interests focus on Networks, Databases, Online learning and Information systems and Sciences.

Shaher Momani received his Ph.D from the university of Wales (UK) in 1991. He then began work at Mutah university in 1991 as assistant professor of applied mathematics and promoted to full Professor in 2006. He left Mutah university to the university of Jordan in 2009 until now. His research interests focus on the numerical solution of fractional differential equations in fluid mechanics, non-Newtonian fluid mechanics and numerical analysis. Prof. Momani has written over 250 research papers and warded several national and international prizes. Also, he was classified as one of the top ten scientists in the world in fractional differential equations according to ISI web of knowledge.