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Stability Criteria for Nonlinear Volterra Integro-Dynamic Systems

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Abstract: We study conditions under which the solutions of nonlinear Volterra integro-dynamic system of the form

$$x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s,x(s))\Delta s$$

are stable on certain time scales. We give sufficient and necessary conditions for various types of stability, including uniform stability, asymptotic stability, exponential asymptotic stability and strong stability.

Keywords: Stability, nonlinear Volterra integro-dynamic, time scale.

1 Introduction and preliminaries

Stability theory is important when examining dynamic responses of a system to disturbances as time approaches infinity [9,10,11,13,16,28]. Stability of nonlinear differential equations or difference equations can be characterized using for example Lyapunov's second method, the method of variation of parameters, inequalities, etc. [4,17,24,25,29,32].

Time scales theory, introduced by Hilger [18] at the end of the twentieth century is a means to unify discrete and differential calculus [5,6]. Volterra and Fredholm type equations (both integral and integro-dynamic) on time scales were discussed in [1,2,3,19,21,22,23,26,27,30,31]. In [26] the authors discuss resolvent asymptotic stability, boundedness of VIDE and show that the principle matrix and resolvent are equivalent for linear VIDE on time scales.

In this paper we provide sufficient conditions for uniform stability, asymptotic stability, exponential asymptotic stability and strong stability of the trivial solution of a nonlinear Volterra intero-dynamic system of the form

$$x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s,x(s))\Delta s,\ x(t_0) = x_0 \quad (1)$$

where $A(t)$ is continuous and a regressive $n\times n$ matrix on $\mathbb{T}_0 := [t_0,\infty)_{\mathbb{T}}, 0 \le t_0 \in \mathbb{T}^k$ and $K(t,s,x)$ is continuous n vector on $\Omega = \{(t,s,x) : t_0 \le s \le t < \infty$ and $x \in \mathbb{R}^n\}$. We obtain new results and we generalize to a time scale some known properties concerning stability from the continuous case [14,20].

In the remainder of this paper we assume that $K(t,s,0) \equiv 0$.

Let $\mathbb{R}^n$ be the space of n-dimensional column vectors $x = \mathrm{col}(x_1,x_2,...x_n)$ with a norm $||\cdot||$. Also, with the same symbol $||\cdot||$ we will denote the corresponding matrix norm in the space $M_n(\mathbb{R})$ of $n\times n$ matrices. If $A \in M_n(\mathbb{R})$, then we denote by A^T its conjugate transpose. We recall that $||A|| := \sup\{||Ax||; ||x|| \le 1\}$ and the following inequality $||Ax|| \le ||A||||x||$ holds for all $A \in M_n(\mathbb{R})$ and $x \in \mathbb{R}^n$.

A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma,\rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$;

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otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$. Given a time scale interval $[a,b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \le t \le b\}$, then $[a,b]_{\mathbb{T}^k}$ denotes the interval $[a,b]_{\mathbb{T}}$ if $a < \rho(b) = b$ and denotes the interval $[a,b)_{\mathbb{T}}$ if $a < \rho(b) < b$. In fact, $[a,b)_{\mathbb{T}} = [a,\rho(b)]_{\mathbb{T}}$. Also, for $a \in \mathbb{T}$, we define $[a,\infty)_{\mathbb{T}} = [a,\infty) \cap \mathbb{T}$. If $\mathbb{T}$ is a bounded time scale, then $\mathbb{T}$ can be identified with $[\inf \mathbb{T}, \sup \mathbb{T}]_{\mathbb{T}}$.

Throughout this work, we assume that $\sup \mathbb{T} = \infty$ with bounded graininess, i.e., $\mu(t) < \infty$. Moreover, the delta derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^k$ is defined by

$$f^{\Delta}(t) = \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

A function f is called rd-continuous provided that it is continuous at right dense points in $\mathbb{T}$, and has finite limit at left-dense points, and the set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$. The set of functions $C^1_{rd}(\mathbb{T},\mathbb{R})$ includes the functions f whose derivative is in $C_{rd}(\mathbb{T},\mathbb{R})$. For $s,t \in \mathbb{T}$ and a function $f \in C_{rd}(\mathbb{T},\mathbb{R})$, the Δ-integral is defined to be

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s),$$

where $F \in C^1_{rd}(\mathbb{T},\mathbb{R})$ is an anti-derivative of f, i.e., $F^{\Delta} = f$ on $\mathbb{T}^k$.

A function $f \in C_{rd}(\mathbb{T},\mathbb{R})$ is called regressive if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$, and $f \in C_{rd}(\mathbb{T},\mathbb{R})$ is called positively regressive if $1 + \mu(t)f(t) > 0$ on $\mathbb{T}^k$. The set of regressive functions and the set of positively regressive functions are denoted by $\mathscr{R}(\mathbb{T},\mathbb{R})$ and $\mathscr{R}^+(\mathbb{T},\mathbb{R})$, respectively.

Let $f \in \mathscr{R}(\mathbb{T},\mathbb{R})$ and $s \in \mathbb{T}$. Then the generalized exponential function $e_f(\cdot,s)$ on a time scale $\mathbb{T}$ is defined to be the unique solution of the initial value problem

$$\begin{cases} x^{\Delta}(t) = f(t)x(t) \\ x(s) = 1. \end{cases}$$

For $h \in \mathbb{R}^+$, set $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -1/h\}$, $\mathbb{Z}_h := \{z \in \mathbb{C} : -\pi/h < \mathrm{Im}(z) \le \pi/h\}$, and $\mathbb{C}_0 := \mathbb{Z}_0 := \mathbb{C}$. For $h \in \mathbb{R}_0^+$ and $z \in \mathbb{C}_h$, the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by

$$\xi_h(z) : = \begin{cases} z, & h = 0 \\ \frac{1}{h}\mathrm{Log}(1+zh), & h > 0, \end{cases}$$

and the exponential function can also be written in the form

$$e_f(t,s) : = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau \right\} \text{ for } s,t \in \mathbb{T}.$$

For more details, see [5]. Clearly, $e_f(t,s)$ never vanishes.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two given time scales and put $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x,y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$, which is a complete metric space with the metric (distance) d defined by

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\text{for } (x_1,y_1),(x_2,y_2) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

A function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is said to be continuous at $(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f(x,y) - f(x_0,y_0)\| < \varepsilon$ for all $(x_0,y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ satisfying $d((x,y),(x_0,y_0)) < \delta$. If (x,y) is an isolated point of $\mathbb{T}_1 \times \mathbb{T}_2$, then the definition implies that every function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is continuous at (x,y). In particular, every function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is continuous at each point of $\mathbb{Z} \times \mathbb{Z}$.

Theorem 1.*([15, Theorem 5 p. 102])Let $f(x,y)$ be a real-finite valued function whose domain is the Cartesian product $S_1 \times S_2$. Suppose $f(x,y)$ is continuous in y at $y = b$ uniformly for x in S_1, and continuous in x at $x = a$ for each y in S_2, then $f(x,y)$ is continuous in (x,y) at (a,b).*

Let $C_{rd}(\mathbb{T}_1 \times \mathbb{T}_2, \mathbb{R})$ denote the set of functions $f(x,y)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ with the following properties:

(i) f is rd-continuous in x for fixed y;

(ii) f is rd-continuous in y for fixed x;

(iii) if $(x_0,y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_0 right-dense or maximal and y_0 right-dense or maximal, then f is continuous at (x_0,y_0);

(iv) if x_0 and y_0 are both left-dense, then the limit of $f(x,y)$ exists (finite) as (x,y) approaches (x_0,y_0) along any path in $\{(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2 : x < x_0, y < y_0\}$.

A brief introduction into the two-variable time scales calculus can be found in [8].

Lemma 1.*([7])*

(i) For a nonnegative φ with $-\varphi \in \mathscr{R}^+$, we have the following inequality

$$1 - \int_s^t \varphi(u)\Delta u \le e_{-\varphi}(t,s) \le \exp\left(-\int_s^t \varphi(u)\Delta u \right)$$

for all $t \ge s$.

(ii) If φ is rd-continuous and non-negative, then

$$1 + \int_s^t \varphi(u)\Delta u \le e_{\varphi}(t,s) \le \exp\left(\int_s^t \varphi(u)\Delta u \right)$$

for all $t \ge s$.

2 Stability

In the remainder of this paper when we say the zero solution of (1) we mean the zero solution of (1) with $x_0 = 0$.

Definition 1. *The zero solution of (1) is stable, if for every $\varepsilon > 0$ there exist a $\delta > 0$ such that for any solution $x(t)$ of (1), the inequality $\|x_0\| < \delta$ implies $\|x(t)\| < \varepsilon$ for $t \in \mathbb{T}_0$.*

In this section, we assume that the zero solution of

$$y^{\Delta}(t) = A(t)y(t),\ y(t_0) = y_0 \tag{2}$$

is stable. This is equivalent [13, Theorem 2.1] to assuming that there exists $\eta > 0$ such that

$$\|\Phi_A(t,s)\| \leq \eta \text{ for } s \in [t_0,t]_{\mathbb{T}}, \tag{3}$$

where $\Phi_A(t,s)$ is a fundamental matrix of (2).

We now put conditions on $K(t,s,x)$ so that the zero solution of (1) is stable.

We make the following assumption:

(A1) There exists $\alpha > 0$ so that $\|K(t,s,x)\| \leq C(t,s)\|x\|$ with $C(t,s)$ rd-continuous for $s \in [t_0,t]_{\mathbb{T}}$ and $\|x\| < \alpha$.

Theorem 2. *Suppose that the assumptions (3) and (A1) hold and there exists a positive constant $M > 0$ such that*

$$\int_{t_0}^{\infty}\int_{t_0}^{s} C(s,u)\Delta u\Delta s < M. \tag{4}$$

Then the zero solution of (1) is stable.

Proof. For any $0 < \varepsilon < \alpha$ let $\delta(\varepsilon) < \frac{\varepsilon}{\eta e^{\eta M}}$ and $\|x_0\| < \delta(\varepsilon)$. Suppose that there exists $t_1 \in \mathbb{T}_0$ such that $\|x(t_1)\| = \varepsilon$ and $\|x(t)\| < \varepsilon$ on $[t_0,t_1)_{\mathbb{T}}$. From the variation of parameters formula [5], we have

$$\begin{aligned}\|x(t)\| &\leq \|\Phi_A(t,t_0)\|\,\|x_0\| + \int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\| \\ &\quad\times \int_{t_0}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s \\ &\leq \eta\delta(\varepsilon) + \eta\int_{t_0}^{t}\int_{t_0}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s\end{aligned}$$

for $t \in [t_0,t_1]_{\mathbb{T}}$.

Let $q(t) = \sup_{s\in[t_0,t]_{\mathbb{T}}} \|x(s)\|$ and we obtain

$$q(t) \leq \eta\delta(\varepsilon) + \eta\int_{t_0}^{t}\int_{t_0}^{s} C(s,u)q(u)\Delta u\Delta s.$$

From Gronwal's inequality [5, Theorem 6.4] and Lemma 1, we have

$$\begin{aligned}\|x(t)\| &\leq q(t) \\ &\leq \eta\delta(\varepsilon)\exp\left(\int_{t_0}^{t}\log\left(\frac{1+\mu(s)\eta\int_{t_0}^{s}C(s,u)\Delta u}{\mu(s)}\right)\Delta s\right) \\ &\leq \eta\delta(\varepsilon)\exp\left(\int_{t_0}^{t}\int_{t_0}^{s}\eta C(s,u)\Delta u\Delta s\right) \\ &\leq \eta\delta(\varepsilon)e^{\eta M} < \varepsilon \text{ for } t \in [t_0,t_1]_{\mathbb{T}}.\end{aligned}$$

Therefore $\|x(t_1)\| < \varepsilon$, which is a contradiction. Thus the zero solution of (1) is stable. The proof is complete.

Instead of (A1) assume

($\tilde{\text{A}}$1) $\|K(t,s,x)\| \leq C(t,s)\|x\|$, where $C(t,s)$ is rd-continuous for $s \in [t_0,t]_{\mathbb{T}}$ and $x \in \mathbb{R}^n$.

Remark. Suppose that the assumptions (3), (4) and ($\tilde{\text{A}}$1) hold. Then the solutions of (1) are bounded.

3 Asymptotic stability

Assume that there exists a constant $\beta > 0$ such that

$$\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\Delta s < \beta \tag{5}$$

for all $t \in \mathbb{T}_0$ with $t \geq \sigma(t_0)$. (This is equivalent [13, Theorem 2.3 and Theorem 2.4] to assuming that the zero solution of (2) is asymptotic stable).

Note that

$$\Phi_A(t,t_0) \to 0 \text{ as } t \to \infty. \tag{6}$$

Definition 2. *The zero solution of (1) is asymptotically stable, if it is stable and attractive (i.e. if for any solution $x(t)$ of (1), there exist $\delta_0 \geq 0$ such that $\|x_0\| < \delta_0$ implies $\|x(t)\| \to 0$ as $t \to \infty$).*

Theorem 3. *Suppose that the assumptions (A1) and (5) hold and*

$$\sup_{t\in\mathbb{T}_0}\int_{t_0}^{t} C(t,s)\Delta s < \frac{1}{\beta}. \tag{7}$$

Furthermore, suppose that

$$\lim_{s\to\infty}\int_{t_0}^{t} C(s,u)\Delta u = 0 \text{ for all } t \in \mathbb{T}_0. \tag{8}$$

Then the zero solution of (1) is asymptotic stable.

Proof. We first show the stability of the zero solution of (1). From (7) there exists a positive constant γ such that

$$0 < \gamma < \frac{1}{\beta} \text{ and } \sup_{s\in\mathbb{T}_0}\int_{t_0}^{s} C(s,u)\Delta u \leq \gamma. \tag{9}$$

From (6) there exists a positive constant N such that

$$\|\Phi_A(t,t_0)\| \leq N \text{ for all } t \in \mathbb{T}_0. \tag{10}$$

For any $0 < \varepsilon < \alpha$ and t_0 let $\delta(\varepsilon) < \min\{(1-\gamma\beta)\varepsilon/N, \varepsilon\}$.

Consider the solution $x(t)$ of (1) such that $\|x_0\| < \delta$. Suppose that there exists $t_1 \in \mathbb{T}_0$ such that $\|x(t_1)\| = \varepsilon$ and $\|x(t)\| < \varepsilon$ on $[t_0,t_1)_{\mathbb{T}_0}$. From the variation of parameters formula [5], we have

$$\begin{aligned}\|x(t)\| &\leq \|\Phi_A(t,t_0)\|\,\|x_0\| + \int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\| \\ &\times \int_{t_0}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s \\ &= \|\Phi_A(t,t_0)\|\,\|x_0\| + \int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\| \\ &\times \int_{t_0}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s \\ &< N\delta + \varepsilon\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\int_{t_0}^{s} C(s,u)\Delta u\Delta s \\ &< (1-\gamma\beta)\varepsilon + \varepsilon\beta\gamma = \varepsilon \text{ for } t\in [t_0,t_1]_{\mathbb{T}}.\end{aligned}$$

Therefore $\|x(t_1)\| < \varepsilon$, which is a contradiction. Thus the zero solution of (1) is stable.

Next we will show that the zero solution of (1) is attractive. Let $\varepsilon = 1$, then there exists $\delta_0 = \delta(1) < 1$ such that $\|x_0\| < \delta_0$ implies

$$\|x(t)\| < \min(\alpha,1) \text{ for all } t\in\mathbb{T}_0. \tag{11}$$

Suppose there exists x_0 with $\|x_0\| < \delta_0$ such that the solution $x(t)$ of (1) satisfies

$$\limsup_{t\to\infty}\|x(t)\| = \lambda > 0. \tag{12}$$

From (9) $\gamma\beta < 1$, and there exists a constant θ such that $\gamma\beta < \theta < 1$. From (12), there exists $t_1\in\mathbb{T}_0$ such that

$$\|x(u)\| \leq \frac{\lambda}{\theta} \text{ for all } u\in[t_1,\infty)_{\mathbb{T}} \tag{13}$$

and from (8), there exists $T\in(t_1,\infty)_{\mathbb{T}}$ such that

$$\int_{t_0}^{t_1} C(s,u)\Delta u < \frac{(\theta-\gamma\beta)\lambda}{2\theta\beta} \text{ for all } s\in[T,\infty)_{\mathbb{T}}. \tag{14}$$

Then we have

$$\begin{aligned}\|x(t)\| &\leq \|\Phi_A(t,t_0)\|\,\delta_0 + \int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\| \\ &\times \int_{t_0}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s \\ &\leq \|\Phi_A(t,t_0)\|\,\delta_0 + \|\Phi_A(t,t_0)\|\int_{t_0}^{T}\|\Phi_A(t_0,\sigma(s))\| \\ &\times \int_{t_0}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s + \int_{T}^{t}\|\Phi_A(t,\sigma(s))\| \\ &\times \int_{t_0}^{t_1} C(s,u)\,\|x(u)\|\,\Delta u\Delta s + \int_{T}^{t}\|\Phi_A(t,\sigma(s))\| \\ &\times \int_{t_1}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s.\end{aligned}$$

From (5), (11) and (14) we have

$$\int_{T}^{t}\|\Phi_A(t,\sigma(s))\|\int_{t_0}^{t_1} C(s,u)\,\|x(u)\|\,\Delta u\Delta s \leq \frac{(\theta-\gamma\beta)\lambda}{2\theta}.$$

Moreover, using (5), (9) and (13), we obtain

$$\int_{T}^{t}\|\Phi_A(t,\sigma(s))\|\int_{t_1}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s \leq \frac{\gamma\beta\lambda}{\theta}.$$

Thus we have

$$\begin{aligned}\|x(t)\| &\leq \|\Phi_A(t,t_0)\|\,\delta_0 + \|\Phi_A(t,t_0)\|\int_{t_0}^{T}\|\Phi_A(t_0,\sigma(s))\| \\ &\times \int_{t_0}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s + \frac{(\theta+\gamma\beta)\lambda}{2\theta}.\end{aligned}$$

Since $\|\Phi_A(t,t_0)\| \to 0$ as $t\to\infty$ by (6), we have $\lambda \leq \frac{(\theta+\gamma\beta)\lambda}{2\theta}$ and thus $\lambda < \lambda$, a contradiction. Therefore the zero solution of (1) is attractive. The proof is complete.

4 Exponential asymptotic stability

Definition 3. *The zero solution of (1) is exponentially asymptotically stable, if there exist $\eta > 0$ and for every $\varepsilon > 0$ there exist $\delta > 0$ such that for any solution $x(t)$ of (1), $\|x_0\| < \delta$ implies $\|x(t)\| < \varepsilon e_{-\eta}(t,t_0)$ for $t\in\mathbb{T}_0$.*

We assume that there exists $M,\eta > 0$ with $-\eta\in\mathscr{R}^+(\mathbb{T},\mathbb{R})$ such that

$$\|\Phi_A(t,s)\| \leq Me_{-\eta}(t,s) \text{ for all } s\in[t_0,t]_{\mathbb{T}}. \tag{15}$$

(This is equivalent [13, Theorem 2.2 and Theorem 2.4] to assuming that the zero solution of (2) is exponentially stable).

Theorem 4. *Suppose that the assumptions (A1) and (15) holds and there exists a positive constant ν such that*

$$\sup_{t\in\mathbb{T}_0}\int_{t_0}^{t} e_{-\nu}(s,\sigma(t))C(t,s)\Delta s < \frac{\eta}{M}. \tag{16}$$

Then the zero solution of (1) is exponentially asymptotically stable.

Proof. Using (15) for all $t\in\mathbb{T}_0$ and $\|x_0\| < \alpha/M$, we have

$$\begin{aligned}\|x(t)\| &\leq \|\Phi_A(t,t_0)\|\,\|x_0\| + \int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\| \\ &\times \int_{t_0}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s \\ &\leq Me_{-\eta}(t,t_0)\,\|x_0\| + M\int_{t_0}^{t} e_{-\eta}(t,\sigma(s)) \\ &\times \int_{t_0}^{s} C(s,u)\,\|x(u)\|\,\Delta u\Delta s.\end{aligned} \tag{17}$$

There exist positive constants $\vartheta < \nu$ and ε with $-\vartheta,-\varepsilon\in\mathscr{R}^+(\mathbb{T},\mathbb{R})$ such that $-\eta = -\vartheta\oplus-\varepsilon$ and

$$\sup_{t\in\mathbb{T}_0}\int_{t_0}^{t} e_{-\vartheta}(s,\sigma(t))C(t,s)\Delta s < \frac{\varepsilon}{M}.$$

Multiply by $e_{-\vartheta}(t_0,t)$ on both sides of (17) to obtain

$$e_{-\vartheta}(t_0,t)\|x(t)\| \le Me_{-\varepsilon}(t,t_0)\|x_0\| + M\int_{t_0}^{t} e_{-\vartheta}(t_0,\sigma(s)) \times e_{-\varepsilon}(t,\sigma(s))\int_{t_0}^{s} C(s,u)\|x(u)\|\Delta u\Delta s$$
$$= Me_{-\varepsilon}(t,t_0)\|x_0\| + M\int_{t_0}^{t} e_{-\varepsilon}(t,\sigma(s))\int_{t_0}^{s} e_{-\vartheta}(u,\sigma(s)) \times C(s,u)e_{-\vartheta}(t_0,u)\|x(u)\|\Delta u\Delta s.$$

If we define $q(t) = \sup_{s\in[t_0,t]_{\mathbb{T}}} e_{-\vartheta}(t_0,s)\|x(s)\|$, it follows that

$$e_{-\vartheta}(t_0,t)\|x(t)\| \le Me_{-\varepsilon}(t,t_0)\|x_0\| + Mq(t)\int_{t_0}^{t} e_{-\varepsilon}(t,\sigma(s))\int_{t_0}^{s} e_{-\vartheta}(u,\sigma(s))C(s,u)\Delta u\Delta s$$
$$\le Me_{-\varepsilon}(t,t_0)\|x_0\| + \varepsilon q(t)\int_{t_0}^{t} e_{-\varepsilon}(t,\sigma(s))\Delta s.$$

Using [5, Theorem 2.39], we obtain that

$$e_{-\vartheta}(t_0,t)\|x(t)\| \le Me_{-\varepsilon}(t,t_0)\|x_0\| + \{1-e_{-\varepsilon}(t,t_0)\}q(t). \tag{18}$$

Now we consider two cases

(I): In this case $e_{-\vartheta}(t_0,s)\|x(s)\| \le e_{-\vartheta}(t_0,t)\|x(t)\|$ for any $s\in[t_0,t]_{\mathbb{T}}$, so we have $q(t) = e_{-\vartheta}(t_0,t)\|x(t)\|$. Then from (18) we have

$$q(t) \le Me_{-\varepsilon}(t,t_0)\|x_0\| + \{1-e_{-\varepsilon}(t,t_0)\}q(t).$$

Thus $q(t) \le M\|x_0\|$ for all $t\in\mathbb{T}_0$. Then $q(t) = e_{-\vartheta}(t_0,t)\|x(t)\|$ implies $\|x(t)\| \le Me_{-\vartheta}(t,t_0)\|x_0\|$ for all $t\in\mathbb{T}_0$.

(II): In this case there exists $s\in[t_0,t]_{\mathbb{T}}$ such that

$$e_{-\vartheta}(t_0,s)\|x(s)\| > e_{-\vartheta}(t_0,t)\|x(t)\|.$$

There exists $t_1\in[t_0,t)_{\mathbb{T}}$ such that $q(t) = e_{-\vartheta}(t_0,t_1)\|x(t_1)\|$. Then from (18) we have

$$q(t_1) = e_{-\vartheta}(t_0,t_1)\|x(t_1)\|$$
$$\le Me_{-\varepsilon}(t_1,t_0)\|x_0\| + \{1-e_{-\varepsilon}(t_1,t_0)\}q(t_1).$$

Thus $q(t_1) \le M\|x_0\|$ for all $t_1\in\mathbb{T}_0$. Then $q(t_1) > e_{-\vartheta}(t_0,t)\|x(t)\|$ implies $\|x(t)\| \le Me_{-\vartheta}(t,t_0)\|x_0\|$ for all $t\in\mathbb{T}_0$.

Thus from (I) and (II), the zero solution of (1) is exponentially asymptotically stable. The proof is complete.

5 Strong stability

Definition 4. *The zero solution of (1) is said to be strongly stable if for every $\varepsilon > 0$, there exist $\delta > 0$ such that, for any solution $x(t)$ of (1), the inequalities $t_1\in\mathbb{T}_0$ and $\|x(t_1)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \ge t_0 \in\mathbb{T}_0$.*

Theorem 5. *([10, Theorem 4.3]) Let $\Phi_A(t,s)$ be a fundamental matrix for (2). Then the zero solution of (2) is strongly stable on $\mathbb{T}_0$ if and only if there exist a positive constant K such that*

$$\left\|\Phi_A(t,t_0)\Phi_A^{-1}(s,t_0)\right\| \le K \text{ for all } t_0\leqslant s\le t<\infty$$

or equivalently,

$$\|\Phi_A(t,t_0)\| \leqslant K \text{ and } \left\|\Phi_A^{-1}(t,t_0)\right\| \leqslant K \text{ for all } t\in\mathbb{T}_0.$$

Let us consider the following hypotheses:

H_1 : There exist a continuous function $\varphi:\mathbb{T}_0\to(0,\infty)$ and the constants $p_1\ge 1, K_1>0$ such that

$$\int_{t_0}^{t}\left(\varphi(s)\left\|\Phi_A(t,t_0)\Phi_A^{-1}(s,t_0)\right\|\right)^{p_1}\Delta s \leqslant K_1, \text{ for all } t\in\mathbb{T}_0.$$

H_2 : There exist a continuous function $\varphi:\mathbb{T}_0\to(0,\infty)$ and the constants $p_2\ge 1, K_2>0$ such that

$$\int_{t_0}^{t}\left(\varphi(s)\left\|\Phi_A^{-1}(t,t_0)\Phi_A(s,t_0)\right\|\right)^{p_2}\Delta s \leqslant K_2, \text{ for all } t\in\mathbb{T}_0.$$

H_3 : There exist a continuous function $\varphi:\mathbb{T}_0\to(0,\infty)$ and the constants $p_3\ge 1, K_3>0$ such that

$$\int_{t_0}^{t}\left(\varphi(s)\left\|\Phi_A^{-1}(s,t_0)\Phi_A(t,t_0)\right\|\right)^{p_3}\Delta s \leqslant K_3, \text{for all } t\in\mathbb{T}_0.$$

H_4 : There exist a continuous function $\varphi:\mathbb{T}_0\to(0,\infty)$ and the constants $p_4\ge 1, K_4>0$ such that

$$\int_{t_0}^{t}\left(\varphi(s)\left\|\Phi_A(s,t_0)\Phi_A^{-1}(t,t_0)\right\|\right)^{p_4}\Delta s \leqslant K_4, \text{ for all } t\in\mathbb{T}_0.$$

Theorem 6. *Suppose that the fundamental matrix $\Phi_A(t,s)$ satisfies one of the following conditions:*

C_1: H_1 and H_2 are true.
C_2: H_1 and H_4 are true.
C_3: H_2 and H_3 are true.
C_4: H_3 and H_4 are true.
Then, the zero solution of (2) is strongly stable on $\mathbb{T}_0$.

Proof. We prove that $\Phi_A(t,t_0)$ and $\Phi_A^{-1}(t,t_0)$ are bounded on $\mathbb{T}_0$. First consider the case C_2. For this we prove that $\Phi_A(t,t_0)$ is bounded on $\mathbb{T}_0$. Consider

$$q(t) = \varphi^{p_1}(t)\left(\|\Phi_A(t,t_0)\|\right)^{-p_1} \text{ for } t\in\mathbb{T}_0.$$

From the identity

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\Phi_A(t,t_0) = \int_{t_0}^{t}\left(\varphi(s)\Phi_A(t,t_0)\Phi_A^{-1}(s,t_0)\right) \times\left(q(s)(\varphi(s))^{-1}\Phi_A(s,t_0)\right)\Delta s, \text{ for } t\in\mathbb{T}_0,$$

it follows that

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\|\Phi_A(t,t_0)\| \le \int_{t_0}^{t}\left(\varphi(s)\left\|\Phi_A(t,t_0)\Phi_A^{-1}(s,t_0)\right\|\right) \times\left(q(s)(\varphi(s))^{-1}\|\Phi_A(s,t_0)\|\right)\Delta s,\ t\in\mathbb{T}_0. \tag{19}$$

If $p_1 = 1$, we have that $q(s)(\varphi(s))^{-1}\|\Phi_A(s,t_0)\| = 1$. From (19) and the hypothesis H_1, it follows that

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\|\Phi_A(t,t_0)\| \le \int_{t_0}^{t} \left(\varphi(s)\left\|\Phi_A(t,t_0)\Phi_A^{-1}(s,t_0)\right\|\right)\Delta s \le K_1,\ t\in\mathbb{T}_0.$$

If $p_1 > 1$, set $q_1 = \frac{p_1}{(p_1-1)}$, such that $q(s)(\varphi(s))^{-1}\|\Phi_A(s,t_0)\| = (q(s))^{1/q_1}$. From (19), it follows that

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\varphi(t)(q(t))^{-1/p_1} \le \int_{t_0}^{t} \left(\varphi(s)\left\|\Phi_A(t,t_0)\Phi_A^{-1}(s,t_0)\right\|\right) \times (q(s))^{1/q_1}\Delta s,\ \text{for } t\in\mathbb{T}_0.$$

Using the Hölder's inequality [5], we obtain

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\varphi(t)(q(t))^{-1/p_1} \le \left(\int_{t_0}^{t} \left(\varphi(s)\left\|\Phi_A(t,t_0)\Phi_A^{-1}(s,t_0)\right\|\right)^{p_1}\Delta s\right)^{1/p_1} \times \left(\int_{t_0}^{t} q(s)\Delta s\right)^{1/q_1} \quad \text{for } t\in\mathbb{T}_0.$$

Now using hypothesis H_1, we obtain

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)^{1/p_1}\varphi(t)(q(t))^{-1/p_1} \le K_1^{1/p_1},\ \text{for } t\in\mathbb{T}_0.$$

or

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\|\Phi_A(t,t_0)\|^{p_1} \le K_1,\ \text{for } t\in\mathbb{T}_0.$$

Thus for $p_1 \ge 1$, the function $\|\Phi_A(t,t_0)\|$ satisfies the inequality

$$\|\Phi_A(t,t_0)\| \le K_1^{1/p_1}\left(\int_{t_0}^{t} q(s)\Delta s\right)^{-1/p_1},\ \text{for } t\in\mathbb{T}_0.$$

Let $Q(t) = \int_{t_0}^{t} q(s)\Delta s$ for $t\in\mathbb{T}_0$, so

$$\|\Phi_A(t,t_0)\| \le K_1^{1/p_1}(Q(t))^{-1/p_1},\ \text{for } t\in\mathbb{T}_0.$$

Note

$$Q^{\Delta}(t) = q(t) \ge K_1^{-1}(\varphi(t))^{p_1}Q(t),\ \text{for } t\in\mathbb{T}_0.$$

It follows that there exist a constant M_1 such that $\|\Phi_A(t,t_0)\| \le M_1$ for $t\in\mathbb{T}_0$.

Now for prove $\Phi_A^{-1}(t,t_0)$ is bounded on $\mathbb{T}_0$. Consider

$$q(t) = \varphi^{p_4}(t)\left\|\Phi_A^{-1}(t,t_0)\right\|^{-p_4} \quad \text{for } t\in\mathbb{T}_0.$$

From the identity

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\Phi_A^{-1}(t,t_0) = \int_{t_0}^{t} \left(q(s)(\varphi(s))^{-1}\Phi_A^{-1}(s,t_0)\right) \times \left(\varphi(s)\Phi_A(s,t_0)\Phi_A^{-1}(t,t_0)\right)\Delta s,\ \text{for } t\in\mathbb{T}_0,$$

it follows that

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\left\|\Phi_A^{-1}(t,t_0)\right\| \le \int_{t_0}^{t} \left(q(s)(\varphi(s))^{-1}\left\|\Phi_A^{-1}(s,t_0)\right\|\right) \times \left(\varphi(s)\left\|\Phi_A(s,t_0)\Phi_A^{-1}(t,t_0)\right\|\right)\Delta s.$$

If $p_4 = 1$, we have that $q(s)(\varphi(s))^{-1}\left\|\Phi_A^{-1}(s,t_0)\right\| = 1$. Using hypothesis H_4 it follows that

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\left\|\Phi_A^{-1}(t,t_0)\right\| \le \int_{t_0}^{t} \left(\varphi(s)\|\Phi_A(s,t_0)\right)\Delta s \times\Phi_A^{-1}(t,t_0)\| \le K_4 \text{ for } t\in\mathbb{T}_0.$$

If $p_4 > 1$, set $q_4 = \frac{p_4}{p_4-1}$, such that $q(s)(\varphi(s))^{-1}\left\|\Phi_A^{-1}(s,t_0)\right\| = (q(s))^{1/q_4}$. It follows that

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\left\|\Phi_A^{-1}(t,t_0)\right\| \le \int_{t_0}^{t} \left(\left\|\Phi_A(s,t_0)\Phi_A^{-1}(s,t_0)\right\|\right) \times (q(s))^{1/q_4}\Delta s \text{ for } t\in\mathbb{T}_0.$$

Using Hölder's inequality, we obtain

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\left\|\Phi_A^{-1}(t,t_0)\right\| \le \left(\int_{t_0}^{t} \left(\varphi(s)\left\|\Phi_A(s,t_0)\Phi_A^{-1}(t,t_0)\right\|\right)^{p_4}\Delta s\right)^{1/p_4} \times \left(\int_{t_0}^{t} q(s)\Delta s\right)^{1/q_4} \quad \text{for } t\in\mathbb{T}_0.$$

Now using hypothesis H_4, we obtain

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)\left\|\Phi_A^{-1}(t,t_0)\right\| \le \left(\int_{t_0}^{t} q(s)\Delta s\right)^{1/q_4} K_4^{1/p_4},\ t\in\mathbb{T}_0$$

or

$$\left(\int_{t_0}^{t} q(s)\Delta s\right)^{1/p_4}\|\Phi_A(t,t_0)\| \le K_4^{1/p_4},\ t\in\mathbb{T}_0.$$

Thus for $p_4 \ge 1$, the function $\left\|\Phi_A^{-1}(t,t_0)\right\|$ satisfies the inequality

$$\left\|\Phi_A^{-1}(t,t_0)\right\| \le K_4^{1/p_4}\left(\int_{t_0}^{t} q(s)\Delta s\right)^{-1/p_4} \quad t\in\mathbb{T}_0.$$

Let $Q(t) = \int_{t_0}^{t} q(s)\Delta s$ for $t\in\mathbb{T}_0$, so

$$\left\|\Phi_A^{-1}(t,t_0)\right\| \le K_4^{1/p_4}(Q(t))^{-1/p_4},\ t\in\mathbb{T}_0.$$

Note $Q^{\Delta}(t) = q(t) \ge K_4^{-1}(\varphi(t))^{p_4}Q(t)$, for $t\in\mathbb{T}_0$. Thus there exist a constant M_2 such that $\left\|\Phi_A^{-1}(t,t_0)\right\| \le M_2$ for $t\in\mathbb{T}_0$.

Hence the conclusion follows immediately from Theorem 5. The proof is similar for the cases C_1,C_3 or C_4.

Our next result gives us an existence and uniqueness criteria for solutions of equation (1).

Theorem 7. *Assume that the function K is continuous and satisfies the condition*

$$\|K(t,s,x)-K(t,s,y)\| \le f(t,s)\|x-y\| \tag{20}$$

for $t_0 \le s \le t < \infty$ and for all $x,y\in\mathbb{R}^n$, and there exist a function $\beta \in \mathscr{R}^+(\mathbb{T}_0,\mathbb{R})$ such that

$$\sup_{t\in\mathbb{T}_0}\frac{1}{e_\beta(t,t_0)}\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\int_{t_0}^{s}f(s,u) \tag{21}$$

$$\times e_\beta(u,t_0)\Delta u\Delta s<1 \tag{22}$$

and

$$\sup_{t\in\mathbb{T}_0}\frac{1}{e_\beta(t,t_0)}\|\Phi_A(t,t_0)\|<\infty, \tag{23}$$

where f is a rd-continuous nonnegative function on $D=\{(t,s):t_0\le s\le t<\infty\}$. Then there exists a unique solution of (1).

Proof. We consider the space of continuous function $C(\mathbb{T}_0;\mathbb{R}^n)$ with

$$\sup_{t\in\mathbb{T}_0}\frac{\|x(t)\|}{e_\beta(t,t_0)}<\infty,$$

and we denote this space by $C_\beta(\mathbb{T}_0;\mathbb{R}^n)$. We couple the linear space $C_\beta(\mathbb{T}_0;\mathbb{R}^n)$ with a metric, namely

$$d_\beta^\infty(x,y)=\sup_{t\in\mathbb{T}_0}\frac{\|x(t)-y(t)\|}{e_\beta(t,t_0)}.$$

It is easy to see that $C_\beta(\mathbb{T}_0;\mathbb{R}^n)$ (coupled with the norm $\|x\|_\beta^\infty=\sup_{t\in\mathbb{T}_0}\frac{\|x(t)\|}{e_\beta(t,t_0)}$) is a Banach space [22, Lemma 4.1].

Consider the operator T from $C_\beta(\mathbb{T}_0;\mathbb{R}^n)$ to $C_\beta(\mathbb{T}_0;\mathbb{R}^n)$ given by

$$Tx(t)=\Phi_A(t,t_0)x(t_0)+\int_{t_0}^{t}\Phi_A(t,\sigma(s))\int_{t_0}^{s}K(s,u,x)\Delta u\Delta s \tag{24}$$

and note

$$\begin{aligned}
\|Tx\|_\beta^\infty &= \sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\Big\|\Phi_A(t,t_0)x(t_0)\\
&+\int_{t_0}^{t}\Phi_A(t,\sigma(s))\int_{t_0}^{s}K(s,u,x)\Delta u\Delta s\Big\|\\
&\le \sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\Big(\|\Phi_A(t,t_0)x(t_0)\|\\
&+\int_{t_0}^{t}\Big\|\Phi_A(t,\sigma(s))\int_{t_0}^{s}K(s,u,x)\Big\|\Delta u\Delta s\Big)\\
&\le \sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\|\Phi_A(t,t_0)\|\,\|x(t_0)\|\\
&+\sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\\
&\times\int_{t_0}^{s}\|K(s,u,x)\|\Delta u\Delta s,
\end{aligned}$$

and using (20), (21) and (23), we obtain

$$\begin{aligned}
\|Tx\|_\beta^\infty &\le \sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\|\Phi_A(t,t_0)\|\,\|x(t_0)\|\\
&+\sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\\
&\times\int_{t_0}^{s}f(s,u)\|x(u)\|\Delta u\Delta s\\
&\le \sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\|\Phi_A(t,t_0)\|\,\|x(t_0)\|\\
&+\sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\\
&\times\int_{t_0}^{s}f(s,u)e_\beta(u,t_0)\tfrac{\|x(u)\|}{e_\beta(u,t_0)}\Delta u\Delta s\\
&\le \sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\|\Phi_A(t,t_0)\|\,\|x(t_0)\|\\
&+\|x\|_\beta^\infty\sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\\
&\times\int_{t_0}^{s}f(s,u)e_\beta(u,t_0)\Delta u\Delta s.
\end{aligned}$$

Also

$$\begin{aligned}
\|Tx-Ty\|_\beta^\infty &= \sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\Big\|\int_{t_0}^{t}\Phi_A(t,\sigma(s))\\
&\times\int_{t_0}^{s}K(s,u,x)-K(s,u,y)\Delta u\Delta s\Big\|\\
&\le \sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\\
&\times\int_{t_0}^{s}\|K(s,u,x)-K(s,u,y)\|\Delta u\Delta s\\
&\le \sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\\
&\times\int_{t_0}^{s}f(s,u)\|x(u)-y(u)\|\Delta u\Delta s\\
&\le \|x-y\|_\beta^\infty\sup_{t\in\mathbb{T}_0}\tfrac{1}{e_\beta(t,t_0)}\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\\
&\times\int_{t_0}^{s}f(s,u)e_\beta(u,t_0)\Delta u\Delta s.
\end{aligned}$$

Hence T is a contraction. The Banach fixed point theorem guarantees there exists a unique solution of the system (1) [22,23](note the variation of parameters formula [5]).

Theorem 8. *Assume that the function K is continuous and satisfies the condition*

$$\|K(t,s,x)-K(t,s,y)\| \le f(t,s)\|x-y\| \tag{25}$$

for $t_0\le s\le t<\infty$ and for all $x,y\in\mathbb{R}^n$, such that

$$\sup_{t\in\mathbb{T}_0}\int_{t_0}^{t}\|\Phi_A(t,\sigma(s))\|\int_{t_0}^{s}f(s,u)\Delta u\Delta s<1 \tag{26}$$

and

$$N=\sup_{t\in\mathbb{T}_0}\|\Phi_A(t,t_0)\|<\infty, \tag{27}$$

where f is a rd-continuous nonnegative function on $D=\{(t,s):t_0\le s\le t<\infty\}$. Then the zero solution of (1) is strongly stable on $\mathbb{T}_0$.

*Proof.*For $t_1 \in \mathbb{T}_0$, equation (24) yields

$$Tx(t) = \Phi_A(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi_A(t,\sigma(s)) \int_{t_0}^{s} K(s,u,x)\Delta u\Delta s$$
$$= \Phi_A(t,t_1)\Phi_A(t_1,t_0)x(t_0) + \int_{t_0}^{t_1} \Phi_A(t,\sigma(s))$$
$$\times \int_{t_0}^{s} K(s,u,x)\Delta u\Delta s + \int_{t_1}^{t} \Phi_A(t,\sigma(s)) \int_{t_0}^{s} K(s,u,x)\Delta u\Delta s$$

$$= \Phi_A(t,t_1)\Phi_A(t_1,t_0)x(t_0) + \int_{t_0}^{t_1} \Phi_A(t,t_1)\Phi_A(t_1,\sigma(s))$$
$$\times \int_{t_0}^{s} K(s,u,x)\Delta u\Delta s + \int_{t_1}^{t} \Phi_A(t,\sigma(s)) \int_{t_0}^{s} K(s,u,x)\Delta u\Delta s$$
$$= \Phi_A(t,t_1)Tx(t_1) + \int_{t_1}^{t} \Phi_A(t,\sigma(s))$$
$$\times \int_{t_0}^{s} K(s,u,x)\Delta u\Delta s.$$

Applying Theorem 7, we deduce that there exists a unique solution $x(t)$ of (1) on $\mathbb{T}_0$, such that

$$\|x(t)\| \le \|\Phi_A(t,t_1)\| \|x(t_1)\| + \int_{t_1}^{t} \|\Phi_A(t,\sigma(s))\|$$
$$\times \int_{t_0}^{s} \|K(s,u,x)\| \Delta u\Delta s$$
$$\le \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \int_{t_1}^{t} \|\Phi_A(t,\sigma(s))\| \int_{t_0}^{s} \|K(s,u,x)\| \Delta u\Delta s$$
$$+ \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \|\Phi_A(t,t_1)\| \|x(t_1)\|.$$

Now using (25), we have

$$\|x(t)\| \le \sup_{t\in\mathbb{T}_0} \|x(t)\| \le \|x(t_1)\| \sup_{t\in\mathbb{T}_0} \|\Phi_A(t,t_0)\|$$
$$+ \sup_{w\in[t_1,\infty)_{\mathbb{T}}} \|x(w)\| \sup_{t\in\mathbb{T}_0} \int_{t_0}^{t} \|\Phi_A(t,\sigma(s))\| \int_{t_0}^{s} f(s,u)\Delta u\Delta s, \tag{28}$$

for $t \ge t_0 \in \mathbb{T}_0$. Let $\varepsilon > 0$ be arbitrary and let $\delta(\varepsilon) = \frac{\varepsilon(1-P)}{N}$, be such that

$$\|x(t_1)\| < \delta(\varepsilon), \tag{29}$$

where

$$P = \sup_{t\in\mathbb{T}_0} \int_{t_0}^{t} \|\Phi_A(t,\sigma(s))\| \int_{t_0}^{s} f(s,u)\Delta u\Delta s.$$

Now (28) yields

$$\sup_{t\in[t_1,\infty)_{\mathbb{T}}} \|x(t)\|(1-P) \le N\|x(t_1)\|, \text{ i.e.,}$$
$$\sup_{t\in[t_1,\infty)_{\mathbb{T}}} \|x(t)\| < \varepsilon.$$

This proves that the zero solution of (1) is strongly stable on $\mathbb{T}_0$. The proof is complete.

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