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On the Inverse Problem for Thermostatted Kinetic Models with Application to the Financial Market

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Abstract: This paper is concerned with the coupling of the inverse problem theory with the thermostatted kinetic theory. Specifically an inverse problem is proposed where the data vector consists of m known measures, the data kernel is a $m \times n$ matrix which depends on the distribution function vector that is solution of the thermostatted kinetic theory model, and the unknown source or signal consists of a n-dimensional vector. In particular the paper focuses on the under-determined inverse problem, namely $m < n$, and the solution is obtained by employing the principle of maximum Shannon entropy of the information theory. Applications refer to the financial market and specifically to the derivation of the information which triggers the evolution of global stock market indexes. Future research directions are also discussed into the last section of the paper.

Keywords: Thermostats, Nonlinearity, Integro-differential Equation, Inverse problem, Shannon entropy

1 Introduction

Since the last century scholars have proposed mathematical models for the analysis of complex phenomena occurring in life sciences systems, including in the finance field. A complex system consists of inhomogeneous and adaptive agents that interact both with each others and the outside environment [1]. The behavior and macroscopic features of the system are the result of the collective nonlinear interactions occurring at the microscopic scale [2–4]. The emerging behavior of the system depends on both the interactions and the ability of the particles to develop specific and autonomous strategies (active particles). Among the complex systems, the financial market has recently attracted many attention.

The first financial market model dates back to Black-Scholes. The model proposed in [5] allows to calculate the value of stock options by considering a risky asset (the stock) and a riskless asset (the cash). In 1982, R. Engle introduces the AutoRegressive Conditional Heteroskedasticity (ARCH) process [6] in order to analyse the financial data series. In statistics, heteroskedasticity is the fact that the local variance of stock prices is a function of time, and the volatility is the measure of this variance. In [7] Bollerslev introduces the Generalized ARCH (GARCH) process, keeping constant the global variance but not the local variance, thus allowing to identify more parameters. Volatility models such as ARCH/GARCH become very popular among traders, considering that volatility is linked to the risk. The reader interested in recent contributions in this field is referred to paper [8] and the references cited therein.

The main aim of this paper is twofold. On the one hand, the paper is concerned with the definition of an inverse problem coupled to the thermostatted kinetic theory. On the other hand, the new framework is proposed for the modeling and analysis of financial markets with special attention to the signals which trigger the market evolution. The arguments of the inverse theory and the thermostatted kinetic theory are thus the main subjects of the present paper.

The inverse theory is an approach well established in the literature [9–11]. A universal formal definition for inverse problems does not exist. However an inverse problem is said ill-posed if there is no solution, or the solution is not unique or unstable, namely arbitrarily small errors in the measurement data may lead to indefinitely large errors in the solution. An inverse problem is called a source problem if it is required to determine the source. The theory of inverse and ill-posed problems is widely used in physics (astronomy, quantum mechanics, acoustics), in geophysics (seismic exploration, electrical, magnetic and

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gravimetric prospecting), medicine (X-ray and NMR tomography), ecology (air and water quality control, space monitoring), economics (optimal control theory, financial mathematics), see [12] and the references cited therein.
The discrete thermostatted kinetic theory has been recently proposed in [13] to model the evolution of a complex system composed of particles which are able to express a strategy. According to the strategy, the particles are divided into different subsystems called functional subsystems. The strategy is modeled by introducing in the microscopic state of the particles a variable, called activity, which can attain discrete real values. The evolution of the system is obtained by employing a statistical approach which is based on the definition of distribution functions. Interactions among the particles include conservative events (changing into the magnitude of the activity variable) and nonconservative events (proliferation and mutation of particles). Differently from the classical framework of the generalized kinetic theory, the thermostatted framework allows to model systems which operate out of equilibrium, namely subjected to external force fields, and in order to ensure the existence of a nonequilibrium stationary state a dissipative term, called thermostat, is introduced. The new thermostatted framework has been employed for the modeling of pedestrian dynamics into a metro station [14]. It is worth stressing that the thermostatted framework generalizes the kinetic approach that has been proposed in [15] for the modeling of vehicular traffic. The thermostatted kinetic theory has been also derived in the case of a continuous activity variable. In particular the continuous thermostatted kinetic theory framework has been employed for the modeling of biological systems, and specifically for the treatment of keloid [16] (thus generalizing the framework proposed in [17]) and for the antigen recognition process by the immune system [18] (see also the recent review paper [19]). The continuous framework has been also investigated for the derivation of macroscopic equations by employing asymptotic limits [20] thus allowing a multiscale analysis.
The present paper deals with applications in finance of the inverse problem theory coupled with the thermostatted kinetic theory. Specifically the interest focuses on the reconstruction of sources/signals which trigger the market evolutions. The inverse problem is treated within the methods of the information theory and in particular the principle of maximum Shannon entropy. In information theory, entropy is the measure of amount of information that is missing before reception. The principle of maximum Shannon entropy states that subject to precisely stated prior data, the probability distribution that best represents the current state of knowledge is the one with largest information theoretical entropy. The reader interested in further details is referred to the review papers [21–24] and the references cited therein. It is worth mentioning that further computable entropy measures have been proposed in the pertinent literature, e.g. the Kolmogorov-Sinai entropy, the Ledrappier-Young Entropy, the Pesin Entropy, the topological entropy, the Von Neumann entropy, see the review paper [25].
The present paper is divided into 5 more sections which follow this introduction. Specifically Section 2 deals with the fundamentals of the discrete thermostatted kinetic theory. Section 3 is devoted to the definition of an inverse problem which is based on the distribution function vector that is solution of the thermostatted kinetic model. Section 4 is concerned with the resolution of the inverse problem by employing the methods of the information theory and specifically the principle of the maximum Shannon entropy. Section 5 focuses with applications and specifically with the derivation of the signals which trigger the time evolution of global stock market indexes. Finally Section 6 concludes the paper with references to further research perspectives.

2 The discrete thermostatted kinetic theory

This section deals with the basics of the discrete thermostatted kinetic theory approach that has been recently proposed in [13] for the modeling of a nonequilibrium complex system. The main aim of this section is to introduce the reader to the underlying mathematical framework which will be coupled with an inverse problem.
Let $\mathscr{S}$ be a complex system composed of interacting particles which are able to express a strategy or function (*active particles*). The system is assumed to be homogeneous with respect to the space and velocity variables. Accordingly, the microscopic state of the particles consists of a scalar variable u, called *activity*, which can attain discrete values, namely

$$u \in D_u = \{u_1, u_2, \dots, u_n\}, \quad u_i \in \mathbb{R}.$$

The time evolution of the system $\mathscr{S}$ is described by employing a statistical mechanics approach which is based on the definition of the following distribution function:

$$f = f(t,u) : [0,+\infty) \times D_u \to \mathbb{R}_+.$$

According to the particle strategy, the overall system $\mathscr{S}$ is divided into $n \in \mathbb{N}$ subsystems called functional subsystems. The time evolution of the i-th functional subsystem, for $i \in \{1,2,...,n\}$, is described by the following distribution function:

$$f_i(t) = f(t,u_i) : [0,+\infty) \to \mathbb{R}_+.$$

Let $\mathbf{f} = \mathbf{f}(t) = (f_1(t), f_2(t), ..., f_n(t)) \in \mathbb{R}^n$ be the distribution function vector. The discrete p-th order moment of the system is defined as follows:

$$\mathbb{E}_p[\mathbf{f}](t) = \sum_{i=1}^{n} u_i^p f_i(t), \quad p \in \mathbb{N}. \tag{1}$$

In particular the density, the linear activity-momentum, and the activity-energy are obtained for $p=0$, $p=1$, and $p=2$, respectively.

The time evolution of the function f_i depends on the particle interactions which consist of jumping into the activity variable values and particle-jumping among the functional subsystems. Specifically the gain particle term $G_i[\mathbf{f}](t)$ and the loss particle term $L_i[\mathbf{f}](t)$ read, respectively:

$$G_i[\mathbf{f}](t) = \sum_{h=1}^{n}\sum_{k=1}^{n} \eta_{hk} A_{hk}^{i} f_h(t) f_k(t), \tag{2}$$

$$L_i[\mathbf{f}](t) = f_i(t) \sum_{k=1}^{n} \eta_{ik} f_k(t), \tag{3}$$

where $\eta_{hk}: D_u^2 \to \mathbb{R}_+$ models the interaction rate between the particle with state u_h and the particle with state u_k; the function $A_{hk}^i = A(u_h,u_k,u_i): D_u^3 \to \mathbb{R}_+$ denotes the probability density that a particle with state u_h falls into a state u_i after an interaction with a particle with state u_k. In particular, the transition function A_{hk}^i has the structure of a probability density with respect to the variable u_i, then:

$$\sum_{i=1}^{n} A_{hk}^{i} = 1, \quad \forall\, h,k \in \{1,2,\dots,n\}. \tag{4}$$

The active particles that are able to change subsystems (jumping subsystem process) can be modeled with the following operator:

$$M_i[\mathbf{f}](t) = \sum_{h=1}^{n}\sum_{k=1}^{n} \eta_{hk}\, \varphi_{hk}^{i}\, f_h(t)\, f_k(t), \tag{5}$$

where φ_{hk}^i is the jumping rate into the i-th subsystem, due to interactions between particles of the h-th subsystem and particles of the k-th subsystem.

Bearing all the above in mind, the time evolution of f_i, for $i \in \{1,2,\dots,n\}$, is described by the following equation:

$$\frac{d f_i}{dt} = J_i[\mathbf{f}] + M_i[\mathbf{f}] = G_i[\mathbf{f}] - L_i[\mathbf{f}] + M_i[\mathbf{f}], \tag{6}$$

which is called *the discrete kinetic theory framework at equilibrium*.
In order to model complex systems out of equilibrium, the external force field acting on the subsystems needs to be defined. Accordingly, let

$$\mathbf{F}(t) = (F_1(t), F_2(t), \dots, F_n(t)) : [0,+\infty[\to \mathbb{R}_+^n$$

be the external force field that maintains the system out of the equilibrium. The time evolution of f_i, for $i \in \{1,2,\dots,n\}$, now reads:

$$\frac{d f_i}{dt} = J_i[\mathbf{f}] + M_i[\mathbf{f}] + F_i - \left(\frac{\mathbf{U}^p\cdot(\mathbf{J}[\mathbf{f}]+\mathbf{F})}{\mathbb{E}_p[\mathbf{f}]}\right), \tag{7}$$

which is called *the discrete thermostatted kinetic theory framework*, where $\mathbf{J}[\mathbf{f}] = (J_1[\mathbf{f}], J_2[\mathbf{f}], \dots, J_n[\mathbf{f}])$ and $\mathbf{U}^p = (u_1^p, u_2^p, \dots, u_n^p)$. In particular the term $-\alpha f_i$, where

$$\alpha(\mathbf{J}[\mathbf{f}], \mathbb{E}_p, \mathbf{F}) = \frac{\mathbf{U}^p\cdot(\mathbf{J}[\mathbf{f}]+\mathbf{F})}{\mathbb{E}_p[\mathbf{f}]} = \frac{\sum_{i=1}^{n} u_i^p (J_i[\mathbf{f}]+F_i)}{\sum_{i=1}^{n} u_i^p f_i} \tag{8}$$

is the dumping term that makes the dynamic dissipative thus avoiding the unbounded increase of the p-th order moment. The term α is called the thermostat term which allows the system to reach a nonequilibrium stationary state in the long-time limit [26–28].

3 The underlying inverse problem

This section is devoted to the inverse problem definition, which is based on the thermostatted kinetic theory framework that has been revised in the previous section. The following main definitions will be used in what follows.

Definition 1.*Let $\mathscr{H}_1$ be the space of the source $s:[0,+\infty)\to\mathbb{R}$ and $\mathscr{H}_2$ be the measurements space (observed data). Let $A:\mathscr{H}_1\to\mathscr{H}_2$ be an operator (data kernel), and $\mu\in\mathscr{H}_2$ an observed data. An inverse problem consists in finding a solution $s\in\mathscr{H}_1$ of the following problem:*

$$\mu = A[s]. \tag{9}$$

Definition 2.*The inverse problem (9) is said well-posed in the Hadamard sense if:*

1.A solution s exists for any μ in the observed data space;
2.The solution s is unique;
3.The inverse mapping $\mu\mapsto s$ is continuous.

This paper is concerned with the following inverse problem. Let $m\in\mathbb{N}^*$ and

$$\mu(t) = (\mu_1(t), \mu_2(t), \dots, \mu_m(t)) : [0,+\infty) \to \mathbb{R}^{m,1}$$

the m-dimensional *data vector*, and

$$\mathbf{K}[\mathbf{f}] : [0,+\infty) \to \mathbb{R}^{m,n}$$

the *data kernel matrix*, which contains the distribution function vector $\mathbf{f}$ solution of the discrete thermostatted kinetic theory framework (7). The present paper is concerned with an inverse problem, which consists in finding the n-dimensional sources (signals) vector

$$s : [0,+\infty) \to \mathbb{R}^{n,1}$$

solution of the following problem:

$$\mu(t) = \mathbf{K}[\mathbf{f}](t)s(t), \tag{10}$$

where

$$\mathbf{K}[\mathbf{f}] = \begin{bmatrix} K_{11}[\mathbf{f}] & K_{12}[\mathbf{f}] & \dots & K_{1n}[\mathbf{f}] \\ K_{21}[\mathbf{f}] & K_{22}[\mathbf{f}] & \dots & K_{2n}[\mathbf{f}] \\ \vdots & & & \vdots \\ K_{m1}[\mathbf{f}] & K_{m2}[\mathbf{f}] & \dots & K_{mn}[\mathbf{f}] \end{bmatrix}, \tag{11}$$

and

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix}, \qquad s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}. \tag{12}$$

It is worth stressing that the inverse problem (10) belongs to the class of the linear inverse problems. In particular the linear inverse problem is well-posed if 1 and 2 hold. If $\mathbf{K}$ is finite-dimensional, the corresponding inverse problem is well-posed if either the property 1 or 2 holds.

Definition 3. *The inverse problem (10) is said under-determined (respectively over-determined) if the number of measures m is less (respectively more) than the number of unknown sources n.*

Example 1. Assume that the complex system is characterized by the following relations:

$$\mu_1[\mathbb{E}_0](t) = \sum_{i=1}^{n} f_i(t) s_i(t), \tag{13}$$

$$\mu_2[\mathbb{E}_2](t) = \sum_{i=1}^{n} u_i^2 f_i(t) s_i(t). \tag{14}$$

Accordingly, the measure μ_1 is related to the density $\mathbb{E}_0$ and the measure μ_2 is related to the activity-energy $\mathbb{E}_2$. Bearing all above in mind, we have:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \qquad \mathbf{K}[\mathbf{f}] = \begin{bmatrix} f_1 & f_2 & \dots & f_n \\ u_1^2 f_1 & u_2^2 f_2 & \dots & u_n^2 f_n \end{bmatrix}, \tag{15}$$

and the unknowns vector reads:

$$s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}. \tag{16}$$

It is worth stressing that if $n = 2$, and the determinant of the matrix $\mathbf{K}[\mathbf{f}]$ is different from zero ($|u_1| \neq |u_2|$ and $f_1(t) \neq 0$, $f_2(t) \neq 0$, $\forall t \in \mathbb{R}_+$) the inverse problem (13-14) admits a unique solution (Cramer system). If $n > 2$ the inverse problem (13) is under-determined.

4 The maximum entropy principle solution

This section is concerned with the problem to find a solution of the inverse problem (10). Specifically the section aims at constructing a solution of the inverse problem (10) in the under-determined case and by employing the maximum Shannon entropy method.

The concept of entropy has been employed in different fields and different type of entropy has been considered, among others, the Clausius thermodynamics entropy [29], the Boltzmann/Gibbs statistical mechanics entropy [30] and the Shannon information theory entropy [31, 32]. In thermodynamics, if a certain small amount of heat δQ is supplied quasi-statically to a system with an absolute temperature T, then the entropy S of the system will increase according to the following relation:

$$dS = \frac{\delta Q}{T}, \tag{17}$$

where d represents an infinitesimal small change of a state function and δ represents that of a path function.
In statistical thermodynamics, the Gibbs thermodynamic entropy S of a thermodynamic system reads:

$$S = -\kappa_B \sum_i p_i \ln p_i, \tag{18}$$

where the summation is taken over each possible state i, p_i is the probability of a microstate and k_B is the Boltzmann constant. In statistical mechanics, the Boltzmann entropy is an approximation of the Gibbs entropy to an ideal gas, namely is obtained under the assumption that all the component particles of a thermodynamic system are statistically independent (the probability distribution of the system as a whole then factorises into the product of N separate identical terms, one term for each particle). Accordingly:

$$S_B = -N\kappa_B \sum_i p_i \ln p_i. \tag{19}$$

In information theory, the Shannon entropy $H(X)$ is a measure of the uncertainty associated with a discrete random variable $X = \{x_1, x_2, \dots, x_n\}$. Specifically $H(X) = \mathbb{E}(I(X))$ where $\mathbb{E}$ is the expected value and I is the information content of X (which is itself a random variable). If p denotes the probability mass function of X then the entropy can explicitly be written as:

$$H(X) = \sum_{i=1}^{n} p(x_i) I(x_i) = -\sum_{i=1}^{n} p(x_i) \log_b p(x_i), \tag{20}$$

where b is the base of the logarithm used.
In [33] Jaynes has derived the principle of maximum entropy from Shannon's expression as a new type of subjective statistic inference to set up probabilistic distributions based on partial knowledge. The principle of maximum entropy, allowing the least biased estimation possible, made entropy a concept independent from mechanical hypotheses and coherent with quantum mechanics.
It is worth stressing that further scientific domains where

the entropy theory has been employed include hydrologic and geomorphic sciences [34], geography [35, 36], economy [37], sociology [38].

As already mentioned, this section describes how to solve the inverse problem (10) by using the maximum Shannon entropy method. As known the information that an event occurs varies inversely with the size of the probability. An event with a very low probability that occurs gives a great deal of information, whereas when an event with a high probability occurs, this gives less information. Accordingly, information varies inversely with probability. Assume that there are two independent events with probabilities p_1 and p_2 where $p_1 + p_2 = 1$. If the two events occur together with probability $p_1 p_2$, the information gained would be proportional to $1/p_1 p_2$, and since the information should be additive, one should have that the information gained is $1/p_1 + 1/p_2$. Accordingly the function I should satisfy the following relation:

$$I\left(\frac{1}{p_1 p_2}\right) = I\left(\frac{1}{p_1}\right) + I\left(\frac{1}{p_2}\right). \tag{21}$$

The only solution of the above equation is $\ln(1/p)$ (information gained by the occurrence of the event or equivalently a measure of the uncertainty of the event occurring). In order to have a value for the overall information for the two events, the expected value can be computed, which reads:

$$H(2) = -p_1 \ln p_1 - p_2 \ln p_2. \tag{22}$$

If the number of events is n and the related probability is p_i, for $i \in \{1,2,\ldots,n\}$, the average information is the expected value of this series, which can be written as follows:

$$H(n) = -\sum_{i=1}^{n} p_i \ln p_i. \tag{23}$$

The function H is the standard information entropy of Shannon [31], which is equivalent to the Boltzmann-Gibbs entropy. The Shannon entropy (23) varies from a minimum value of zero to a maximum value of $\ln(n)$. Specifically when $H = 0$, then one event dominates, that is, $p_i = 1$ and $p_k = 0$ for all $k \neq i$; when $H = \ln(n)$, then $p_i = 1/n$ for all i.

The maximum Shannon entropy method consists in choosing a distribution that is the most likely or probable within the constraints, because it is easy to show that the maximum entropy is an approximation to the probability of a particular macrostate occurring among all possible arrangements (or microstates) of the considered events.

Let $\mu \in \mathbb{R}_+$. Bearing all above in mind, the inverse problem consists in finding n sources s_i under the following constraints:

$$C_1)\ \sum_{i=1}^{n} s_i = 1, \tag{24}$$

$$C_2)\ \mu = \sum_{i=1}^{n} K_i[\mathbf{f}] s_i, \tag{25}$$

with $\mathbf{f}$ solution of the discrete thermostatted kinetic theory framework (7). Accordingly the lagrangian function $\mathscr{L}[\mathbf{f}]$ reads:

$$\begin{aligned}\mathscr{L}[\mathbf{f}](s_i,\lambda_0,\lambda_1) = &-\sum_{i=1}^{n} s_i \ln s_i - (\lambda_0 - 1)\left(\sum_{i=1}^{n} s_i - 1\right) \\ &-\lambda_1\left(\mu - \sum_{i=1}^{n} K_i[\mathbf{f}] s_i\right),\end{aligned} \tag{26}$$

where $(\lambda_0 - 1)$ and λ_1 are the related Lagrangian multipliers. Differentiating the lagrangian function $\mathscr{L}[\mathbf{f}]$ with respect to each source s_i and setting the result equals to zero yields:

$$\frac{\partial \mathscr{L}[\mathbf{f}]}{\partial s_i} = -\ln s_i - \lambda_0 - \lambda_1 K_i[\mathbf{f}] = 0, \quad i \in \{1,2,\ldots,n\}. \tag{27}$$

Accordingly the probability model reads:

$$s_i[\mathbf{f}](t) = \exp(-\lambda_0 - \lambda_1 K_i[\mathbf{f}]), \quad i \in \{1,2,\ldots,n\}. \tag{28}$$

The above formula explains why the first multiplier has been set $\lambda_0 - 1$. The values of the parameters λ_0 and λ_1 can be determined by solving the model according to the constraint equations. In particular:

$$\exp(\lambda_0) = \sum_{i=1}^{n} \exp(-\lambda_1 K_i[\mathbf{f}]).$$

The exponential model can be rewritten as follows:

$$s_i[\mathbf{f}](t) = \frac{\exp(-\lambda_1 K_i[\mathbf{f}])}{\sum_{i=1}^{n} \exp(-\lambda_1 K_i[\mathbf{f}])}, \quad \sum_{i=1}^{n} s_i = 1. \tag{29}$$

It is worth stressing that if the Lagrangian multiplier $\lambda_1 = 0$ then the exponential model collapses to a uniform distribution where $s_i = 1/n$. In particular the entropy for this model is at its maximum when

$$H_{\max} = \sum_{i=1}^{n} s_i \ln(\exp(-\lambda_0 - \lambda_1 K_i[\mathbf{f}])) = \lambda_0 + \lambda_1 \mu.$$

This maximum is a function of each multiplier and its constraint, with the implication that entropy is a function of the spread of the distribution, which is determined by the constraint.

Bearing all above in mind, the generalized problem can be solved. Specifically, let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)^T$ be the m-dimensional measure vector, $s = (s_1, s_2, \ldots, s_n)^T$ the n-dimensional source vector, with $m < n$ (under-determined system). The maximum Shannon entropy method allows to find the general model for the source s_i:

$$\begin{cases} s_1 = \dfrac{1}{Z[\mathbf{f}](\lambda)} \exp\left(-\sum_{i=1}^{m} \lambda_i K_{i1}[\mathbf{f}]\right) \\ s_2 = \dfrac{1}{Z[\mathbf{f}](\lambda)} \exp\left(-\sum_{i=1}^{m} \lambda_i K_{i2}[\mathbf{f}]\right) \\ \vdots \\ s_n = \dfrac{1}{Z[\mathbf{f}](\lambda)} \exp\left(-\sum_{i=1}^{m} \lambda_i K_{in}[\mathbf{f}]\right) \end{cases} \quad (30)$$

where $Z[\mathbf{f}]$ is the partition function that reads:

$$Z[\mathbf{f}](\lambda) = \sum_{j=1}^{n} \exp\left(-\sum_{i=1}^{m} \lambda_i K_{ij}[\mathbf{f}]\right), \quad (31)$$

and λ_i is the Lagrange multiplier obtained by replacing the source s_i in the equation (10), namely the vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is solution of the following problem:

$$\nabla(\ln(Z[\mathbf{f}](\lambda))) = -\mu, \quad (32)$$

and then

$$\begin{cases} \frac{\partial \ln(Z[\mathbf{f}](\lambda))}{\partial \lambda_1} + \mu_1 = 0 \\ \frac{\partial \ln(Z[\mathbf{f}](\lambda))}{\partial \lambda_2} + \mu_2 = 0 \\ \vdots \\ \frac{\partial \ln(Z[\mathbf{f}](\lambda))}{\partial \lambda_m} + \mu_m = 0 \end{cases} \quad (33)$$

5 Applications: The financial market

The thermostatted kinetic theory framework coupled with the inverse problem theory can be considered as a general paradigm for the modeling of complex systems. In particular an important application is related to the financial market which is composed by traders. A trader denotes an individual who buys and sells financial items, e.g. stocks, bonds, metals, agricultural products. The price reflect supply and demand thus information defines the market evolution.

Bearing the new framework in mind, the system $\mathscr{S}$ under consideration is the international market, which is composed by three markets (functional subsystems): New York (S_1), London (S_2), and Paris (S_3), where the activity variable represents the trading ability, see the table 1. The system is modeled according to the following main assumptions:

A_1) The system operates at equilibrium, then $F_i = 0$, for all $i \in \{1,2,3\}$.

A_2) The interaction rate η_{ij}, for $i, j \in \{1,2,3\}$, is constant.

A_3) The traders do not change their regional market, then $\varphi^i_{hk} = 0$, for all $i,h,k \in \{1,2,3\}$.

Table 1: The functional subsystems, the activity variable, and the distribution functions.

Subsystems	Activity	Distribution function
New York	Trading ability	$f_1(t)$
London	Trading ability	$f_2(t)$
Paris	Trading ability	$f_3(t)$

The table of games $\mathscr{A}^i_{hk}$ is derived according to a leader-follower dynamics. Specifically:

• If $h = k$, namely the traders h and k have the same ability, the interaction does not imply a change in their ability, then:

$$\mathscr{A}^i_{hh} = \begin{cases} 1 & \text{if } i = h \\ 0 & \text{otherwise} \end{cases} \quad \forall h \in \{1,2,3\}. \quad (34)$$

• The dollar has the biggest influence on the other markets considering that it is the reference currency. Accordingly the subsystems S_2 and S_3 tends to follow S_1. The table of games is thus defined as follows:

$$\mathscr{A}^i_{12} = \mathscr{A}^i_{21} = \begin{cases} 1-\alpha & \text{if } i = 1 \\ \alpha & \text{if } i = 2 \\ 0 & \text{if } i = 3 \end{cases} \quad (35)$$

and

$$\mathscr{A}^i_{13} = \mathscr{A}^i_{31} = \begin{cases} 1-\beta & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ \beta & \text{if } i = 3 \end{cases} \quad (36)$$

where $0 < \alpha, \beta < 1$.

• The Brexit appears as an important factor for the evolution of the subsystem S_2 considering that it is expected that the London market can plummet. Accordingly the table of games $\mathscr{A}^i_{32}$ and $\mathscr{A}^i_{23}$ follow S_2 describing a significant decrease in the market evolution:

$$\mathscr{A}^i_{23} = \mathscr{A}^i_{32} = \begin{cases} 0 & \text{if } i = 1 \\ 1-\gamma & \text{if } i = 2 \\ \gamma & \text{if } i = 3 \end{cases} \quad (37)$$

where $0 < \gamma < 1$.

Bearing all the above in mind, the evolution equations of the model read:

$$\begin{aligned} \frac{df_1}{dt}(t) &= (1-\alpha)\eta_{21} f_2(t) f_1(t) + (1-\beta)\eta_{31} f_3(t) f_1(t) \\ &\quad -\alpha\eta_{12} f_1(t) f_2(t) - \beta\eta_{13} f_1(t) f_3(t), \\ \frac{df_2}{dt}(t) &= \alpha\eta_{12} f_1(t) f_2(t) + (1-\gamma)\eta_{32} f_3(t) f_2(t) \\ &\quad -(1-\alpha)\eta_{21} f_2(t) f_1(t) - \gamma\eta_{23} f_2(t) f_3(t), \\ \frac{df_3}{dt}(t) &= \beta\eta_{13} f_1(t) f_3(t) + \gamma\eta_{23} f_2(t) f_3(t) \\ &\quad -(1-\beta)\eta_{31} f_3(t) f_1(t) - (1-\gamma)\eta_{32} f_3(t) f_2(t). \end{aligned}$$

The inverse problem is now defined. The information which triggers or modifies the market evolution is assumed to be a source/signal $s(t) = (s_1(t), s_2(t), s_3(t))$, where each s_i refers to the market i and such that $s_1(t) + s_2(t) + s_3(t) = 1$. The measure μ is related to the *stock market index*, which tracks a portfolio of stocks. Specifically μ is a global stock market index, such as the *MSCI World* or the *S&P Global* 100, which includes stocks from multiple markets. Accordingly the evolution of μ is assumed depends on the dynamics among the three financial markets as follows:

$$\mu(t) = \sum_{i=1}^{3} K_i[\mathbf{f}](t) s_i(t), \tag{38}$$

where $K_i[\mathbf{f}](t) = f_i(t)$ and $\mathbf{f} = (f_1, f_2, f_3)$ is solution of the above defined model. Specifically the inverse problem reads:

$$\mu(t) = f_1(t)s_1(t) + f_2(t)s_2(t) + f_3(t)s_3(t).$$

The solution of the inverse problem is thus obtained by employing (30) and it reads:

$$s_i[\mathbf{f}](t) = \exp(-\lambda_0(t) - \lambda_1(t) f_i(t)), \quad i \in \{1,2,3\}, \tag{39}$$

where (λ_0, λ_1) is solution of the following problem:

$$\begin{cases} \lambda_0(t) = \ln\left(e^{-\lambda_1(t) f_1(t)} + e^{-\lambda_1(t) f_2(t)} + e^{-\lambda_1(t) f_3(t)}\right), \\ \sum_{i=1}^{3} f_i \exp(-\lambda_0(t) - \lambda_1(t) f_i(t)) = \mu(t). \end{cases}$$

A numerical analysis can be performed for the quantitative resolution of the inverse problem and for a future tuning of the model with the empirical data.

6 Research perspectives

The present paper has been devoted to a further generalization of the discrete thermostatted kinetic theory proposed in [13] in order to resolve inverse problems which are set within this framework. The inverse problem has been resolved by employing the methods of the information theory and specifically the principle of the maximum Shannon entropy. The solution is based on a probabilistic approach considering that the unknown source/signal is assumed to be a discrete random variable vector.

Future research directions can be pursued from the theoretical and applications point of views. Indeed the Shannon entropy is not the most general measure that can be employed. In fact the Shannon entropy does not appear an appropriated measure when the events are not independents and if the entropy phase space does not allow probabilistic events to occur in all parts of the space. In particular, if the degree of information can be improved, an *a priori* distribution function of the sources can be defined and the relative entropy method can be applied [39–42]. Precisely if the prior distribution is denoted by $\{q_i\}$, then the information (also known as the discrete Kullback-Leibler divergence) reads:

$$I = \sum_{i=1}^{n} p_i \ln\left(\frac{p_i}{q_i}\right).$$

In this context, the distribution $\{p_i\}$ is called the posterior distribution. I takes values between zero and infinity, and it is zero when $p_i = q_i$, $\forall i$, which means that no difference exists between prior and posterior distributions (no information is gained by moving from the prior to the posterior). It is worth stressing that the concept of discrete entropy presents some restrictions, including the fact that depends on n (see [24]), then in some applications appear more suitable the use of a continuous random variable and thus a continuous distribution function. Accordingly the inverse problem needs to be defined within the framework of the continuous thermostatted kinetic theory, where the activity variable $u \in D_u \subseteq \mathbb{R}$ is continuous, see [16, 18]. In this case each distribution function writes $f(t,u)$ and the inverse problem now reads:

$$\mu(t) = \int_0^t K[\mathbf{f}](t,u)s(u)du\ ,\ t \in [0,+\infty), \tag{40}$$

where

– $\mu(t) = (\mu_1(t), \mu_2(t), \ldots, \mu_m(t)) : [0,+\infty) \to \mathbb{R}^{m,1}$ is the m-dimensional data vector, $m \in \mathbb{N}^*$;
– $s : D_u \to \mathbb{R}^{n,1}$ is the n-dimensional sources vector, $n \in \mathbb{N}^*$;
– $K[\mathbf{f}](t,u) : [0,+\infty) \times D_u \to \mathbb{R}^{m,n}$ is the data kernel matrix (Green's function), which contains the distribution functions vector solution of the continuous thermostatted framework.

The continuous inverse problem (40) belongs to the class of Volterra integral equations of the first kind [9]. The reader interested to some algorithms of resolution is referred to the book [10], the paper [43] and the references cited therein. However, the solution of the inverse problem (40) can be investigated in the context of the information theory by means of the definition of a continuous Shannon entropy measure and a continuous relative entropy. This investigation constitutes the basis of future works.

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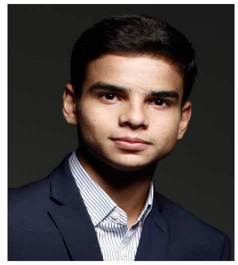

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