Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110524

Wehrl Entropy, Entropy Squeezing and Nonlocal Correlation of Moving Atoms in Squeezed Coherent Field

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Received: 1 Jun. 2017, Revised: 2 July. 2017, Accepted: 4 Aug. 2017
Published online: 1 Sep. 2017

Abstract: We discuss the time-dependent interaction between moving two two-level atoms and squeezed coherent field. We obtain the wave function in the presence of time dependent coupling between the squeezed coherent field and two-two level atoms. Some statistical and nonclassical properties of the squeezed coherent field are discussed through the evolution of the Wehrl entropy. The effects of the initial atomic state position, squeeze parameter and shape function of time-dependent coupling within the atomic speed are examined. It is shown that the atomic speed and squeeze have the potential effect on the time evolution of the entanglement, the Wehrl entropy and the single entropy squeezing. Finally, the results clarified that the manipulation of atom-atom entanglement and the entanglement between the two atoms and squeezed field are greatly controlled by a suitable choice of the squeeze parameter and time dependent coupling among the two atoms and squeezed field.

Keywords: Atom-atom entanglement; Wehrl entropy; single entropy squeezing; two moving atoms; squeezed coherent field.

1 Introduction

It is well known that the dynamical properties of the quantum systems can be treated by Wehrl entropy (WE) [1,2,3]. In this way, WE have been successfully applied in description of different properties of the quantum optical fields such as phase-space uncertainty [4], etc. Also, the problem of measuring quantum correlations (entanglement) in phase space with application of the WE has been discussed [5]. The effect of phase damping of the classical correlation measured by WE and Wehrl phase distribution (Wehrl PD) has been investigated [6]. Also, the nonclassicality of the fields based on the evolution of Wehrl entropy with and without atomic motion effect has discussed [7].

Entanglement is a type of nonlocal correlation that has been playing an important role in the field of quantum information processing, precisely engineered entangled states of interest can indeed be both fragile and difficult to manufacture. It is often viewed as a fragile and exotic feature of quantum mechanics, and its investigation of practical and theoretical significance. Interestingly, entanglement is a property of nonlocal correlations between two or more quantum systems, which cannot be increased under local operations and classical communications [8]. Exploiting its features, it is used as an essential resource for information processing tasks such as quantum computation, quantum teleportation [9], superdense coding [10], quantum cryptography [11,12] and more recently, one-way quantum computation [13] and quantum metrology [14]. These different quantum applications cannot be performed by classical resources and they are based on entangled states.

The quantification of entanglement is necessary to understand and develop the quantum information theory. For this reason different entanglement measures have been used for the mixed and pure states such as concurrence [15,16] entanglement of formation [17], and negativity [18,19]. Also, the concurrence and negativity are used as a good entanglement measures for mixed states, but the von Neumann entropy has been proposed for pure state entanglement [20], all these measures to test whether a given quantum state is separable or entangled.

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Also, some interesting physical phenomena are observed as a result of entanglement measure, such as ”entanglement sudden death” (ESD), entanglement sudden birth (ESB) [21,22].

Considering the motion of the atoms, the Tavis-Jaynes-Cummings Model (TJCM) with two moving atoms has been investigated [23] and the authors of [24] have shown the ESD and ESB which also experimentally observed for entangled photon pairs [25] and the atom ensembles [26]. In addition the entanglement and geometric phase of two moving two-level atom interacting with the field initially prepared in a coherent state has been investigated [27], where the results show that, the atomic motion in terms of the atomic speed and acceleration play a central and impact role in the dynamics of the atom-atom entanglement and geometric phase. More recently the effect of time dependent coupling on the dynamical properties of the nonlocal correlation between two three-level atoms has been examined [31]. The results have shown that the entanglement between the two atoms decreases by increasing the photons multiplicity when the time dependent coupling effect is ignored.

The paper is prepared in the following order: In section II the analytical solution for the model and density matrix will be presented. The Wehrl entropy, single atom entropy squeezing and entanglement quantifiers in terms of the field and atomic density matrix will be defined in sections III, and IV. In section V we will discuss the numerical results. The main conclusion will be summarized in section VI.

2 Model and its dynamics

We consider the model of the time-dependent interaction between the input field mode F and two moving two-level atoms (TLAs) A, B with energy levels denoted by $|g\rangle_j$ is the lower level and $|e\rangle_j$ is the upper level of j^{th} atom ($j = A, B$). The interaction Hamiltonian $\hat{H}_I$ of the system in the rotating-wave approximation (RWA) can be written as [27]

$$\hat{H}_I(t) = \sum_{j=1}^{2} \xi_j(t)\ (\hat{a}^{\dagger}\hat{S}_{-}^{(j)} + \hat{a}\hat{S}_{+}^{(j)}). \tag{1}$$

where $\hat{a}$ $(\hat{a}^{\dagger})$ the annihilation (creation) operator of the field mode while $\hat{S}_{+}^{(j)}(\hat{S}_{-}^{(j)})$ is the usual raising (lowering) for j^{th} two-level atom. Also, $\xi_j(t)$ is the shape function of the cavity field which describe the one-dimensional atomic motion [28,29]. Here, we consider the symmetric case where the moving TLAs have the same shape function of time dependent coupling

$$\xi_1(t) = \xi_2(t) = \varepsilon \sin(\beta t + \delta), \tag{2}$$

where β is the atomic speed and (ε, δ) are an arbitrary constants. For the case in which $\beta = 0$ and $\delta = \pi/2$, the constant coupling between the field and TLAs [30].

We assume that the initial state of the whole system is $|\psi(0)\rangle = |\psi_{AB}(0)\rangle \otimes |\psi_F(0)\rangle$ where $|\psi_{AB}(0)\rangle$ is the initial general state of the TLAs (i.e.$|\psi_{AB}(0)\rangle = (z_1|ee\rangle + z_2|eg\rangle + z_3|ge\rangle + z_4|gg\rangle)$) and $|\psi_F(0)\rangle$ is the initial state of the field which supposed to be in the squeezed coherent state

$$|\psi_F(0)\rangle = |\alpha, r\rangle = \sum_{n=0}^{\infty} b_n |n\rangle, \tag{3}$$

where

$$b_n = \sqrt{\frac{(\frac{1}{2}\tanh(r))^n}{n!\cosh(r)}} \exp\left\{[\tanh(r) - 1]\frac{\alpha^2}{2}\right\} \times H_n\left(\frac{\alpha}{\sqrt{\sinh(2r)}}\right), \tag{4}$$

with r is the squeeze parameter.

The wave function $|\psi(t)\rangle$ at any time $t > 0$, takes the form

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} b_n \Big\{ F_1(n,t)|ee,n\rangle + F_2(n,t)|eg,n+1\rangle F_3(n,t)|ge,n+1\rangle + F_4(n,t)|gg,n+2\rangle \Big\}. \tag{5}$$

The coefficients $F_j(n,t)$, $j = 1,2,3,4$ are obtained by solving the Schrödinger equation $(i\ \partial|\psi(t)\rangle/\partial t = \hat{H}_I(t)|\psi(t)\rangle)$. For example when the TLAs are initially in Bell states $F_j(n,t)$ have the form

$$\begin{aligned}
F_1(n,t) &= \frac{1}{\sqrt{2}}\cos\left(\lambda f(t)\sqrt{4n+6}\right) + \frac{\sqrt{n!\,(n+2)!}}{[n!(n+2)+(n+1)!]\sqrt{2}} \times \left\{\cos\left(f(t)\sqrt{4n+6}\right) - 1\right\}, \\
F_2(n,t) &= L_3(n,t) = -i\left\{\sqrt{\frac{n+1}{8n+12}} + \sqrt{\frac{n+2}{8n+12}}\right\} \times \sin\left(f(t)\sqrt{4n+6}\right), \\
F_4(n,t) &= \frac{1}{n!(2n+3)\sqrt{2}}\Big\{n!\left[n+1+(n+2)\cos\left(f(t)\sqrt{4n+6}\right)\right] + \sqrt{n!(n+2)!}\left[\cos\left(f(t)\sqrt{4n+6}\right) - 1\right]\Big\}.
\end{aligned}$$

where

$$f(t) = \int_0^t \xi(\tau)d\tau. \tag{6}$$

The atomic density matrix $\hat{\rho}_{AB}(t)$ can be calculated by taking the trace of the field basis as follows

$$\hat{\rho}_{AB}(t) = Tr_F(|\psi(t)\rangle\langle\psi(t)|) = \sum_{j=1}^{4}\sum_{k=1}^{4}\rho_{jk}(t)|j\rangle\langle k|, \quad (7)$$

where

$$\rho_{jj}(t) = \sum_{n=0}^{\infty}\left|b_n F_j(n,t)\right|^2, \quad j = 1,2,3,4 \quad (8)$$

are the atomic occupation probabilities of the atomic basis $|ee\rangle, |eg\rangle, |ge\rangle$ and $|gg\rangle$, respectively, and the other elements of the atomic density matrix are given by

$$\rho_{12} = \rho_{13} = \sum_{n=0}^{\infty} b_n b_{n+1} F_1(n+1,t)F_2^*(n,t), \quad (9)$$

$$\rho_{14} = \sum_{n=0}^{\infty} b_n b_{n+2} F_1(n+2,t)F_4^*(n,t), \quad (10)$$

$$\rho_{23} = \sum_{n=0}^{\infty} |b_n F_2(n,t)|^2, \quad (11)$$

$$\rho_{24} = \sum_{n=0}^{\infty} b_n b_{n+1} F_2(n+1,t)F_4^*(n,t), \quad (12)$$

$$\rho_{34} = \sum_{n=0}^{\infty} b_n b_{n+1} F_3(n+1,t)F_4^*(n,t), \; \rho_{kj} = \rho_{jk}^*. \quad (13)$$

On the other hand the field density matrix

$$\begin{aligned}\rho_F(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_n b_m \Big\{ & F_1(n,t)F_1^*(m,t)|n\rangle\langle m| \\ & +2F_2(n,t)F_2^*(m,t)|n+1\rangle\langle m+1| \\ & +F_4(n,t)F_4^*(m,t)|n+2\rangle\langle m+2| \Big\}. \quad (14)\end{aligned}$$

In the following section, we use the relations obtained above to define the single atom entropy squeezing, entanglement and Wehrl entropy for the system under consideration.

3 Wehrl entropy (WE) and single-atom entropy squeezing (SAES)

There are some measures that can be applied in the classical phase space. In this regard, WE is a very informative measure describing the time evolution of a quantum system [32,33,34]. WE was introduced as a classical entropy of a quantum state and can give additional insights into the dynamics of the system, as compared to other entropies. Any quantum state, described by a density matrix ρ_F, can be represented by the Husimi quasi-distribution function, $Q_{\rho_F} = 1/\pi\langle\upsilon|\rho_F(t)|\upsilon\rangle$, where $|\upsilon\rangle$ is the coherent state. The Q-function in the ϖ space is defined as follows

$$Q_{\varpi}(t) = \frac{1}{\pi}\langle\varpi|\hat{\rho}_F(t)|\varpi\rangle \quad (15)$$

Now we use the definition of WE as a quantifier of the statistical properties of a quantum state of the field $\hat{\rho}_F$ defined as [34].

$$S_W(t) = -\int_0^{2\pi}\int_0^{\infty} Q_{\varpi}(t)\ln Q_{\varpi}(t)|\varpi|d|\varpi|d\vartheta \quad (16)$$

where $\varpi = |\varpi|\exp(i\vartheta)$.

The single atom entropy squeezing is consider as application to the evolution of the reduced density matrix operator of atom $\rho_A(t) = Tr_B\{\rho_{AB}(t)\}$ [35,36]. Let S_X, S_Y and S_Z are the atomic operators of the reduced atomic density operator$\rho_A(t)$. So the information entropies in terms of S_X, S_Y and S_Z has the form [35,36]

$$H(S_k) = -\sum_{\ell=0}^{1}\left\{\frac{1}{2}+(-1)^{\ell}\langle S_k\rangle\right\}\ln\left\{\frac{1}{2}+(-1)^{\ell}\langle S_k\rangle\right\},$$
and $k = X,Y,Z.$ (17)

The fluctuation of the component S_k for $k = X$ or Y of the atomic dipole is said to be ”squeezed in entropy” if the information entropy $H(S_k)$ of S_k satisfies the condition, $E(S_k) = \exp(H(S_k)) - \sqrt{\frac{2}{\exp(H(S_Z))}}$. Therefore $E(S_X)$ and $E(S_Y)$ quantify the single atom entropy squeezing in the component S_X and S_Y respectively.

4 Entanglement quantifier

In this section we use the von Neumann entropy to measure the entanglement between TLAs and squeezed coherent field. The expression of the von Neumann entropy takes the form [38]

$$S = -Tr(\hat{\rho}\ln\hat{\rho}) \quad (18)$$

This is zero for all pure states $\hat{\rho}^2 = \hat{\rho}$, where $\hat{\rho}$ is the density operator describing a given quantum state. For the system under consideration i.e. $\hat{\rho} = \hat{\rho}_{AB}$ and the von Neumann entropy can be written as [39,40]

$$S_{AB}(t) = -\sum_{j=1}^{4}\eta_j(t)\ln\eta_j(t), \quad (19)$$

where $\eta_j(t)$ are the eigenvalues of $\hat{\rho}_{AB}(t)$.

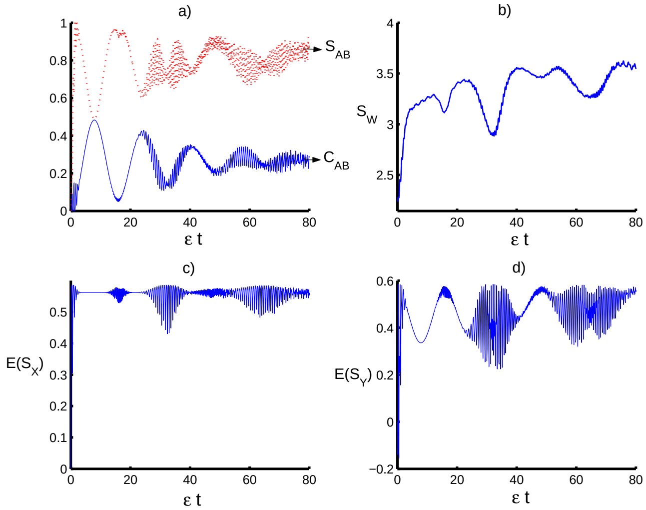

Fig. 1: The time evolution of the: (a) concurrence C_{AB} (solid line), von Neumann entropy S_{AB} (dashed line), (b) Wehrl entropy S_W, (c,d) the SAES components $E(S_X)$, $E(S_Y)$ of a TLAs interacting with field initially in the coherent states (i.e. for $r=0$) for $\alpha=5$. The TLAs are initially in the upper states (i.e. $z_1=1, z_2=z_3=z_4=0$) and constant coupling case (i.e. $\beta=0$, $\delta=\pi/2$).

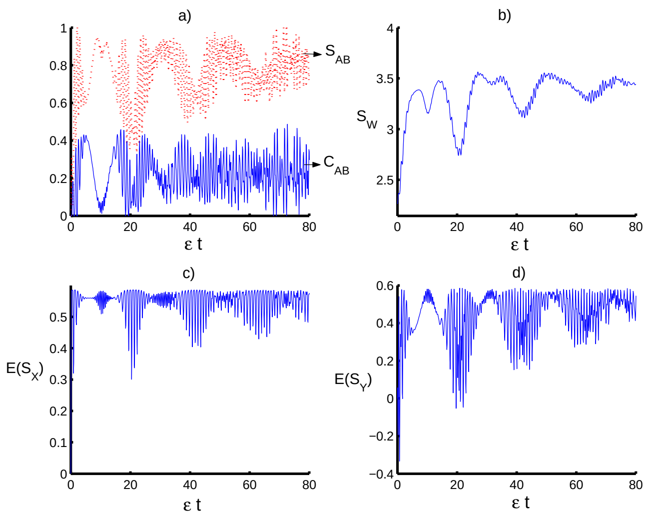

Fig. 2: The same as Fig.1 but the field is initially in the squeezed coherent states for $r=0.5$ and $\alpha=5$.

It is well known that the concurrence is considered as the optimal measure to quantify the atom-atom entanglement. The concurrence of the two atoms based on the atomic density matrix $\hat{\rho}_{AB}(t)$, is given by [16]

$$C_{AB}(t) = \max\{0, \sqrt{\mu_1} - \sqrt{\mu_2} - \sqrt{\mu_3} - \sqrt{\mu_4}\}, \quad (20)$$

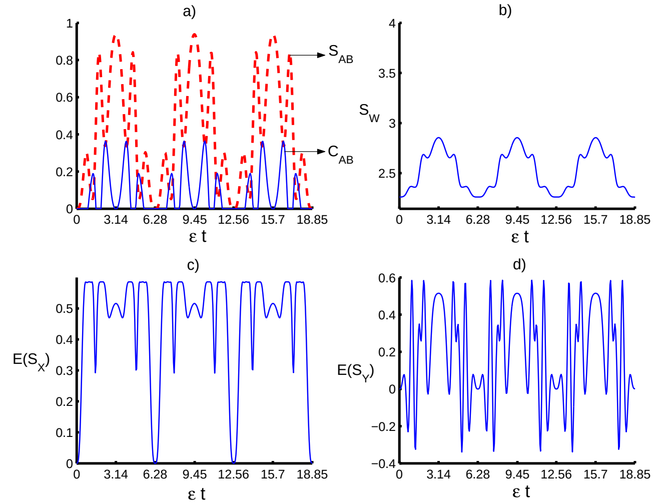

Fig. 3: Effect of symmetric time dependent coupling (i.e. $\xi_1(t) = \xi_2(t) = \varepsilon\sin(t)$ on the time evolution of the: (a) concurrence C_{AB} (solid line), von Neumann entropy S_{AB} (dashed line), (b) Wehrl entropy S_W, (c,d) the SAES components $E(S_X)$, $E(S_Y)$ of a TLAs interacting with field initially in the squeezed coherent states (i.e. for $r=0.5$) for $\alpha=5$. The TLAs are initially in the upper states (i.e. $z_1=1, z_2=z_3=z_4=0$) and constant coupling case (i.e. $\beta=0$, $\delta=\pi/2$).

where μ_i are the eigenvalues of the non-Hermitian matrix $\hat{\rho}_{AB}(t)\tilde{\rho}(t)$ and listed in decreasing order of $\hat{\rho}_{AB}(t)\tilde{\rho}(t)$, while $\tilde{\rho}(t)$ is the spin-flipped state of density operator $\hat{\rho}_{AB}(t)$

$$\tilde{\rho}(t) = (\hat{\sigma}_y \otimes \hat{\sigma}_y)(\hat{\rho}_{AB}(t))^*(\hat{\sigma}_y \otimes \hat{\sigma}_y), \quad (21)$$

where $(\hat{\rho}_{AB}(t))^*$ is conjugate of $\hat{\rho}_{AB}(t)$ in the standard basis of two qubits and $\hat{\sigma}_y$ is the Pauli spin operator. The concurrence has zero value i.e. $C_{AB}(t)=0$ for separable state, whereas $C_{AB}(t)=1$ for the Bell states.

5 Numerical results and discussion

In Figs. 1-6 we discuss the main results of the dynamical properties of the concurrence as a measure of atom-atom entanglement E(A-B), von Neumann entropy as measure of the entanglement between the TLAs and squeezed field E(AB-F). In addition to the dynamical behavior of the SAES components $E(S_X)$ and $E(S_Y)$. The results obtained by exploiting the effect of the physical parameters on the quantum quantifiers and showing the required main conditions for obtaining a high amount of E(A-B), and E(AB-F). A reasonable comparison between the coherent and squeezed coherent field states will enable us to understand the contribution of this effect of

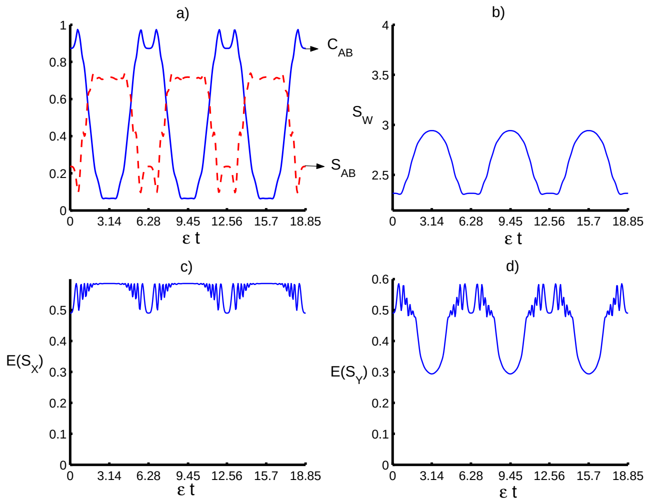

Fig. 4: The same as Figure 3 but the TLAs are initially taken in the Bell states.

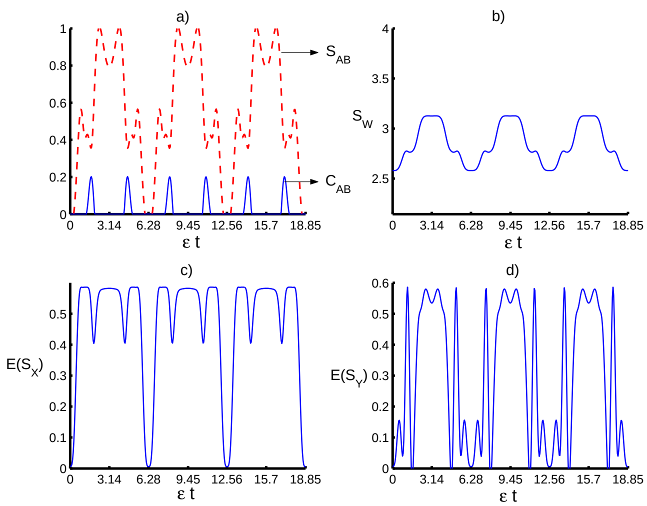

Fig. 5: Effect of strong squeezing regime ($r = 1$) on the time evolution of the: (a) concurrence C_{AB} (solid line), von Neumann entropy S_{AB} (dashed line), (b) Wehrl entropy S_W, (c,d) the SAES components $E(S_X)$, $E(S_Y)$ of a TLAs interacting with field initially in the squeezed coherent states (i.e. for $r = 0.5$) for $\alpha = 5$. The TLAs are initially in the upper states (i.e. $z_1 = 1, z_2 = z_3 = z_4 = 0$) and constant coupling case (i.e. $\beta = 0$, $\delta = \pi/2$).

the initial field state preparation on the dynamical properties of quantum quantifiers. In Figs. 1(a), 1(c), and 1(d) we display, respectively, the variation of as a measure of E(AB-F), as a measure of E(A-B), the SAES components $E(S_X)$ and $E(S_Y)$. It is observed that there is an opposite monotone behavior between $S_{AB}(t)$ and C_{AB} which lead to the decreasing for the entanglement

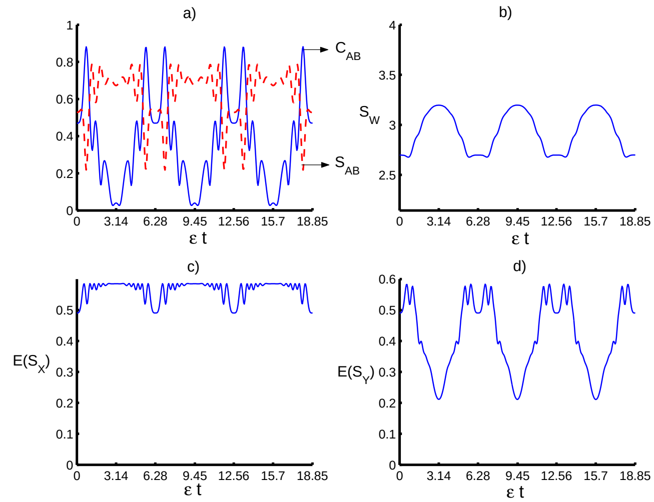

Fig. 6: The same as Figure 5 but TLAs are initially taken in the maximally in Bell states.

between the two atoms. The maximum values of the entanglement quantifiers, increases gradually during time evolution and the oscillations become irregular when the scaled time is significantly large. In other words, when the system is in a chaotic state and it loses its purity, then it becomes partially mixed (see Fig. 1(a)). The maximum E(AB-F) is obtained during the time evolution. There is no squeezing at all in the component $E(S_X)$ where the squeezing only appears in the component $E(S_Y)$ at the initial stage of the time evolution. The WE is shown in Fig. 1(b). It oscillates between a maximum and minimum values which indicates the field close to the quantum and classical states respectively. The WE shows monotonically increasing behavior. This behavior is due to the diffusion in phase space of the Q-function as the time develops. Interestingly, the evolution of the WE presents a richer information about the dynamical properties of the interaction between the TLAs and coherent field.

Figure 2 depicts the effect of squeeze parameter on the evolution of the statistical quantities. As seen all the quantities are affected by the change of the initial state of the field from the coherent state ($r = 0$) to the squeeze coherent state ($r = 0.5$). The entanglement is enhanced in the case of squeezed field. The field being more quantum as the time goes on and the intensity of oscillations of all quantities increase in a chaotic behavior.

In Fig 3 we study the effect of the symmetric time dependent coupling within the atomic speed on the dynamics of the statistical quantities. It is observed that E(AB-F) is enhanced with reducing of entanglement between the TLAs E(A-B). Interestingly the maxima

E(AB-F) corresponds to the death of E(A-B) and zero value of $E(S_X)$ at $\varepsilon t = (2m+1)\pi$ where $m = 0,1,2,...$ Also, the maximum value of S_W is obtained at these points where the field being more quantum. Therefore, we have a clear connection between the entanglement of the system and statistical properties of the field. From Figs. (c, d) we see the link between the entanglement, non-classicality, and squeezing where the squeezing only occurring in the components $E(S_Y)$ at the same time.

To visualize the effect of the initial state setting on the dynamical properties of the entanglement, entropy squeezing and non-classical properties so we plot in figure 4 the time evolution of all quantities when the two-atom start the interaction from the Bell states. As seen from the comparison between figures 3 and 4 the initial atomic state of the TLAs has a significant role in the dynamics on the E(A-B), E(AB-F), WE and entropy squeezing. It is clear that the E(A-B) increases with appearance of the long living entanglement between the TLAs. Also, there is no squeezing at all in the components $E(S_X)$ and $E(S_Y)$. On the other hand the properties of the field are affected by the Bell state where the maximum atom-atom entanglement corresponding to the maximum WE which the maximum entanglement between the squeezed field and TLAs corresponding the field close to classical.

In order to study the impact role of the squeeze parameter on the temporal behavior of the information quantifiers. In this regard, we plot the quantities as a function of the scaled time for when the TLAs initial in the upper state (Fig. 5) and the Bell state (Fig. 6) the main difference between Figs (3,4) and (5,6) can be observed in the two points. Firstly, the E(AB-F) increases and E(A-B) decreases where the TLAs starts from the upper state which E(A-B) is largely increased where the TLAs initially in Bell states. Secondly, the increasing of the squeeze parameter does not have a clear effect on the WE and entropy squeezing. From the above analysis a high amount of squeezing and entanglement of the system under consideration can be obtained by optimal choice of the initial atomic state setting and squeeze parameter

6 Conclusion

In summary, we have introduced a mathematical model describing the interaction between a two two-level atom and field mode initially prepared in the squeezed coherent state. The proposed model is used to perform different tasks of quantum information and computation. We have obtained the solution of Schrodinger equation when the time dependent coupling between the field and two atoms is considered. We have investigated the evolution of some measures such as Wehrl entropy which was used as a measure of non-classicality when the field initially defined in coherent and squeezed coherent states. The von Neumann entropy was used as a measure of the nonlocal correlation between the two atoms and squeezed coherent field while the atom-atom entanglement has been measured by the concurrence. We have shown that the preservation and enhancement of the nonlocal correlation greatly benefit with respect to the atomic state position and squeeze parameter. Furthermore, we have found that the entanglement between two atoms with field can be controlled by the time dependent coupling. Also, we have found that the field that are far from the classical case when the in the presence of atom-atom entanglement while the field is close to the classical case when the field is maximally entangled with the two atoms. Our results provide a useful quantum system to combat the influence of squeeze parameter on the nonlocal correlation by a proper choice of the physical parameters involved in the system under consideration.

Acknowledgement

The authors are grateful to the Agency of Graduate Studies and Scientific Research College of Science, Princess Nourah University, Ministry of Higher Education of Saudi Arabia for financial support of this work (project no. 37/200).

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