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Numerical Solution of a Forward-Backward Equation From Physiology

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Abstract: The aim of this paper is to determine the numerical solution of an equation which models the nerve conduction in a myelinated axon. An appropriate stimulus begins a propagate action potential which travels down the axon. It can be understood as a traveling wave of voltage. It is proposed a computational approximation for the solution of a forward-backward differential equation that models nerve conduction. We look for a solution of an equation defined in $\mathbb{R}$, which tends to known values at $\pm\infty$. Extending the approach introduced in [13, 29, 14] for linear case, a numerical method for the solution of problem, adapted to non linear case, is described. Numerical results using a test problem and a continuation method are computed and analyzed.

Keywords: Mixed-type functional differential equations, non linear boundary value problem, nerve conduction, continuation method, numerical approximation, method of steps

1 Introduction

The main goal is to model nerve conduction in a myelinated[1] axon. Accordingly with the author of [31], nerve fibers transmit information encoded as uniform impulses, denominated the action potentials. These impulses consists in brief changes of membrane polarization. In the majority of nerves and skeletal muscle membranes, an impulse results from a transient flow of Na^+ into the fiber followed by an outflow of K^+. The flow of these ions is down their respective and opposite the electrochemical gradient. This movement of ions takes place in discrete membrane sites.

In [2] is presented some work about behavior of models of myelinated axons. The same author, in [3], studies some details about a diffusive model for a myelinated axon. In particular, this work takes into account a nonlinear mixed type functional differential equation (MTFDE) with deviating arguments from nerve conduction theory that describes the potential propagation along a myelinated nerve axon, where the membrane has a coat of myelin (figure 1) with spaced holes denominated the nodes of Ranvier.

In the mathematical modeling of nerve conduction it is necessary to solve numerically a nonlinear delayed-advanced differential equation of the type

$$x'(t) = F(x(t)) + \beta(t)x(t-\tau) + \gamma(t)x(t+\tau), \qquad (1)$$

where x is the unknown function, β, γ and F are known functions. τ is some positive constant.

At second half of 20 century, the author of [17] made an important contribution for the analysis of MTFDE's in the optimal control theory. Later, in [18, 19] there is a good contribution about functional analysis of linear autonomous MTFDE's.

More recently, in [1], the approximation of a mixed type functional differential equation is done by transforming the original problem in a boundary value problem (BVP).

The main interest is the development, extension and adaptation of numerical methods which solve numerically equation (1), when $F(x(t)) = \alpha(t)x(t)$ and α, β and γ are smooth functions of t and τ is known. These numerical schemes were introduced initially for a linear autonomous

[1] Myelin is a mix of proteins and phospholipids creating an insulating sheath which involves many nerve fibers, raising the speed at which impulses are propagated.

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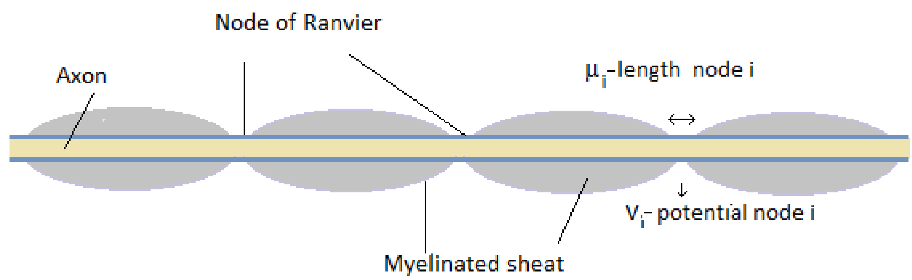

Fig. 1: Myelinated nerve axon.

case in [29] and extended to a non-autonomous case at in [13], using collocation. The numerical approach for linear case was further developed in [14], where a numerical scheme, using the finite element method, was proposed for the solution of such boundary value problems. The authors of [9] focused on the decomposition, by numerical methods, of solutions to MTFDE's into sums of forward solutions and backward solutions. In [28] it is solved a nonlinear MTFDE which models human phonation.

In particular, the equation in study is given by (2)

$$RCv'(t) = F(v(t)) + v(t-\tau) + v(t+\tau), \quad -\infty < t < +\infty. \tag{2}$$

The equation (2) is a boundary value problem (BVP) of first order. We look for for a solution defined in $\mathbb{R}$, which satisfies the nonlinear MTFDE (2) with boundary conditions (3)

$$\begin{cases} v(-\infty) = 0, \\ v(+\infty) = 1. \end{cases} \tag{3}$$

These boundary conditions correspond to the rest and the maximum activated potential, respectively.

The unknown function $v(t)$ represents the transmembrane action potential at a node in a myelinated axon, considering the nerve conduction model. F is related with the current-voltage model as it will be discussed below. R and C are respectively the axomatic nodal resistivity and the nodal capacity. τ is the inverse of the wave potential speed propagation down the axon, it is unknown. A detailed derivation of the model (2)-(3) can be found in [6]. This mathematical model is formulated from an equivalent electric circuit model which assumes pure saltatory conduction. When compared with the membrane, myelin has higher resistance and lower capacitance. If the membrane is depolarized[2] at a node, the action potential[3] tends to jump to the next node and excite the membrane there. The nerve impulse travels the axon one way only, until reach the axon terminal (place where signals in a axon link with other axons).

It is supposed that nodes have the same length μ_i and are equally spaced and electrically similar, the potential cross-sectional variations in axon are negligible and the axon is infinite in extent.

When it is used an adequate variable substitution, equation (2) can be reduced to the following non dimensional model:

$$v'(t) = f(v(t)) + v(t-\tau) + v(t+\tau) - 2v(t), \quad t \in \mathbb{R}, \tag{4}$$

where τ is the non dimensional time delay.

Several models can be obtained using different current-voltage expressions. Here it is used the FitzHugh-Nagumo dynamics for the nodal membrane, without a recovery term and it is assumed that a supra-threshold stimulus begins a propagated axon potential and consequently travels down the axon from node to node. This can be interpreted as a traveling wave of voltage just like it is presented in next section. The function f is given by

$$f(v) = bv(v-a)(1-v), \tag{5}$$

where a is the threshold potential in the non dimensional problem $(0 < a < 1)$ and b is a parameter related with the strength of the ionic current density $(b > 0)$. The solution at any node should be monotone increasing. This arises from the current-voltage relation $f(v)$: once a node is turned on, it cannot return to the rest potential $v = 0$.

The equation (4) is autonomous, therefore if $v(t)$ is a solution of the problem (4)-(3), then any function of the form $v(t+\sigma)$, with $\sigma \in \mathbb{R}$, will also be a solution. To specify a particular solution to this problem, we need to impose an additional condition. Following the authors of [6], we will set

$$v(0) = 0.5 \tag{6}$$

[2] A membrane is depolarized when there is a negative change of membrane potential.

[3] In physiology, the action potential in neurons are often named nerve impulse. It is a set of electrical response due an appropriate stimulus. It is a short-time occurrence where the membrane potential increases and decreases rapidly.

to guarantee the uniqueness of solution.

In this article we have continued the numerical investigation of a nonlinear MTFDE started in [15]. This work is an extended version of [25, 23]. In next section there are presented some details about the formulation of mathematical model, the asymptotic behavior of solution, as well as a test problem with known solutions so we can test the proposed method. It is also described how to apply the method of steps. In sections 3 and 4 is described the numerical scheme to determine the solution. In last two sections are presented and discussed the results and obtained some conclusions.

2 Preliminaries

In this section we will present and discuss some properties of BVP (4)-(3) which will be needed for its numerical solution.

This is not an easy problem to solve numerically: the equation contains both retarded and advanced arguments and the deviation τ is unknown. Moreover, boundary conditions are given at infinity.

We shall begin by introduce some details about the model, followed by the asymptotic analysis of the solutions of the problem (4)-(3). Then, we will present an alternative approach to compute a numerical solution, starting from an asymptotic expansion.

2.1 The model

Summarizing, nervous propagation in a axon is a natural process which results from combination of two phenomena: in myelinated region, a very fast propagation and a quickly reduction of signal strength, in the nodes, a slow propagation and a growth of signal strength chemically stimulated. There exists a threshold behavior, some conditions lead to a decay solution, others to a non-decay solution (sub-threshold or supra-threshold response). From electrical and chemical analysis, at node i we have the potential v_i at each node i, given by equation (7), as it was presented in [12]

$$C\dot{v}_i = \frac{1}{\mu_i LRp}\{v_{i+1}+v_{i-1}-2v_i\} - I_{ionic}(v_i), \quad i \in \mathbb{Z}, \quad (7)$$

where I_{ionic} is the current-voltage relation, usually a bistable non-linear function which satisfy some conditions, μ_i is the length of node i, R is the resistance per unit length, p is the perimeter of the fibre, C is the capacitance, L is the length of myelinated sheath between nodes. In [6, 7] current-voltage relation is given by cubic polynomial function. Equation (7) is a particular case of model introduced by Bell in [2], when we discard the recovery term.

In [3], the sufficient condition $v_i(+\infty) = 1,\ i \in \mathbb{Z}$ guarantees that all nodes are entirely activated.

If we consider that nodes are uniformly spaced and the signal propagates an constant speed, we get

$$v_{i+1}(t) = v_i(t-\tau), \quad i \in \mathbb{Z}. \quad (8)$$

When we introduce (8) in lattice equations (7) we get a MTFDE with the form (2), the discrete Fitzhugh-Nagumo equation, where the delay τ corresponds to the time the signal reaches the closest neighbor node which depends on the distance between nodes and the propagation speed of impulse; $v(t)$, $v(t\pm\tau)$ are the potential at a node and neighboring nodes respectively.

We can see easily that the solution of (2) as the form of a travel wave solution.

In [7], a discrete traveling wave can be formulated by equations (9) and (10)

$$\begin{cases} -c\dot{\varphi}(\xi) = \alpha L_n\varphi(\xi) - g(\varphi(\xi)), & \xi \in \mathbb{R}, \\ \varphi(-\infty) = 0, \qquad \varphi(+\infty) = 1, \end{cases} \quad (9)$$

with

$$L_n\varphi(\xi) = \{\textstyle\sum_{k=1}^{n}\varphi(\xi+\sigma_k)+\varphi(\xi-\sigma_k)\} - 2n\varphi(\xi), \quad (10)$$

where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, g is a non linear function, α some positive scalar. Typically, g is a bistable third degree polynomial function.

For a detailed description of the solution of equation (7) as a traveling wave of voltage, we can see the work developed in [10].

A spatially discrete problem similar to the one-dimensional partial differential equation, the reaction-diffusion equation, can be formulated in following differential-difference equation

$$u_t(x,t) = \alpha\,(u(x+1,t) - 2u(x,t) + u(x-1,t)) - f(u,a), \quad u : \mathbb{Z}\times\mathbb{R} \longrightarrow \mathbb{R}, \quad (11)$$

with $\alpha > 0$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ as a bistable nonlinear third degree function

$$f(u,a) = u(u-a)(u-1), \qquad 0 < a < 1,$$

where a a detuning parameter. Equation (11) define a countable system of ODE's indexed by integer x on a spatial lattice. When we take the one-dimensional traveling wave $u(x,t) = \varphi(x-ct)$ and substitute in (11) we get

$$-c\varphi'(\xi) = \alpha\,(\varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1)) - f(\varphi(\xi),a), \quad (12)$$

which assume the forms (9) and (10) when $n = 1$.

Considering constant node lengths, the substitution $v_i(t) = \varphi(i-ct)$ in equation (7) provides the traveling wave solution with the form (12). In this differential difference equation, we impose that $\varphi(0) = a$ and $\varphi(-\infty) = 0$, $\varphi(+\infty) = 1$. The potential of impulse propagation from (7) can be considered a traveling wave solution. Some details about the global structure of traveling waves can be found in [16].

2.2 Asymptotic behavior of solution

In problem (4)-(3) f is $C^1[0,1]$ and verifies $f(0)=f(1)=0$, $f'(0)<0$ and $f'(1)<0$. The deviation τ and the monotone increasing solution $v(t)$, satisfying $0<v(t)<1$ are computed at the same time.

The study of the asymptotic behavior of solution at $-\infty$ and $+\infty$ and on right is essential to proceed and implement the numerical scheme. We follow an approach similar to the one considered in [6].

Consider $t\le -L$, with L being a positive constant.

When $t\to-\infty$, we have $v(t)\to 0$ and $f(v(t))\to 0$. Thus, we can use the Taylor expansion of f for x close to zero, $f(x)=a_1x+a_2x^2+a_3x^3+O(x^4)$. Introducing this expansion in (4) we obtain

$$\begin{aligned} v'(t) &= a_1v(t)+a_2v(t)^2+a_3v(t)^3+v(t-\tau)\\ &\quad +v(t+\tau)-2v(t)+O(v(t)^4),\\ &= a_2v(t)^2+a_3v(t)^3+v(t-\tau)+v(t+\tau)\\ &\quad +(a_1-2)v(t)+O(v(t)^4),\\ &= a_2v(t)^2+a_3v(t)^3+L_\tau v(t)+O(v(t)^4), \end{aligned} \tag{13}$$

where

$$L_\tau v(t)=v(t-\tau)+v(t+\tau)+(a_1-2)v(t),$$

with $-\infty<t\le -L$, $a_1=f'(0)$, $a_2=f''(0)/2$, $a_3=f'''(0)/3!$.

Let $\varepsilon=v(-L)$; then $\varepsilon\longrightarrow 0$ when $L\to+\infty$. Moreover, let us search for v in the form of a series of powers of ε:

$$v(t)=\varepsilon u_1(t)+\varepsilon^2u_2(t)+\varepsilon^3u_3(t)+O(\varepsilon^4), \tag{14}$$

where u_1, u_2 and u_3 satisfy the initial conditions

$$\begin{cases} u_1(-L)=1,\\ u_2(-L)=0,\\ u_3(-L)=0. \end{cases} \tag{15}$$

By replacing (14) in (13), we conclude that the coefficients u_1, u_2 and u_3 of this series satisfy the system of equations

$$\begin{cases} u_1'(t)-L_\tau u_1(t)=0,\\ u_2'(t)-L_\tau u_2(t)=a_2u_1(t)^2,\\ u_3'(t)-L_\tau u_3(t)=a_3u_1(t)^3+2a_2u_1(t)u_2(t). \end{cases} \tag{16}$$

Since the first equation of system (16) is linear and homogeneous, we obtain the characteristic equation

$$\lambda+2-a_1-2cosh(\lambda\tau)=0. \tag{17}$$

Knowing that $a_1=f'(0)<0$, equation (17) has exactly two roots, one positive λ_+ and one negative (figures 2, 3). Since $u_1(t)=Ce^{\lambda t}$, where C is a constant, we choose the positive root so that $u_1(t)\longrightarrow 0$ as $t\to-\infty$.

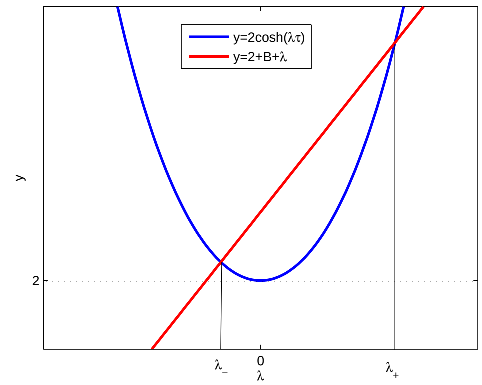

Fig. 2: Roots of the Characteristic Equation $\lambda+2+B-2cosh(\lambda\tau)=0$: roots λ_- and λ_+. $B=-a_1>0$ (in the case of equation (17)); $B=A_1>0$ (in the case of equation (24)).

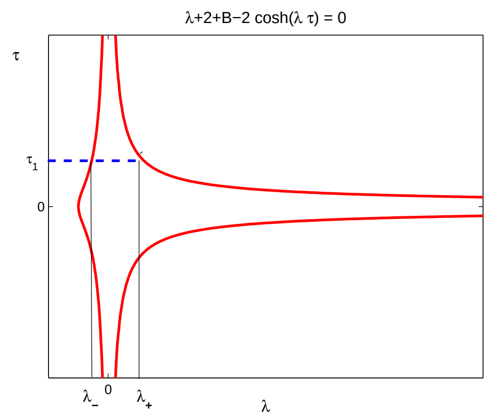

Fig. 3: Characteristic Equation: roots λ_- and λ_+ for a specific value of τ ($\tau=\tau_1$). $B=-a_1>0$ (in the case of equation (17)); $B=A_1>0$ (in the case of equation (24)).

After some computations taking into consideration the initial conditions (15), when $t\le -L$ the asymptotic expansion (14) takes the form

$$\begin{aligned} v(t) &= \varepsilon e^{\lambda_+(t+L)}+\varepsilon^2b_1(e^{2\lambda_+(t+L)}-e^{\lambda_+(t+L)})\\ &\quad +\varepsilon^3(b_2e^{2\lambda_+(t+L)}+b_3e^{3\lambda_+(t+L)}-(b_2+b_3)e^{\lambda_+(t+L)})\\ &\quad +O(\varepsilon^4). \end{aligned} \tag{18}$$

Notice that $\varepsilon=v(-L)$, b_1, b_2, b_3 and λ_+ are constants depending on the Taylor expansion of f and characteristic equation (17). Namely, the $b's$ are given by

$$\begin{aligned} b_1 &= \frac{a_2}{2\lambda_+-a_1+2-2cosh(\lambda_+\tau)},\\ b_2 &= -2b_1^2,\\ b_3 &= \frac{2a_1b_1+a_3}{3\lambda_+-a_1+2-2cosh(3\lambda_+\tau)}. \end{aligned} \tag{19}$$

Consider $t\ge L$, where L is a positive constant.

When $t \to +\infty$, we have $v(t) \to 1$ and $f(v(t)) \to 0$. Writing the Taylor expansion of $f(x)$ as $x \to 1$, we have $f(x) = A_1(1-x) + A_2(1-x)^2 + A_3(1-x)^3 + O((1-x)^4)$, where $A_1 = -f'(1)$, $A_2 = -f''(1)/2$, $A_3 = -f'''(1)/3!$.

Substituting in (4), we obtain

$$\begin{aligned} v'(t) &= -A_1v(t) - A_2v(t)^2 - A_3v(t)^3 + v(t-\tau) \\ &\quad + v(t+\tau) - 2v(t) + O(v(t)^4), \\ &= -A_2v(t)^2 - A_3v(t)^3 + v(t-\tau) + v(t+\tau) \\ &\quad -(A_1+2)v(t) + O(v(t)^4), \\ &= -A_2v(t)^2 - A_3v(t)^3 + K_\tau v(t) + O(v(t)^4), \end{aligned} \tag{20}$$

with $L \le t < +\infty$, where K_τ is defined by

$$K_\tau v(t) = v(t-\tau) + v(t+\tau) - (A_1+2)v(t).$$

In this case, we search for v in the form:

$$v(t) = \varepsilon_+ w_1(t) + \varepsilon_+^2 w_2(t) + \varepsilon_+^3 w_3(t) + O(\varepsilon_+^4), \tag{21}$$

where w_1, w_2 and w_3 satisfy

$$\begin{cases} w_1(L) = 1, \\ w_2(L) = 0, \\ w_3(L) = 0. \end{cases} \tag{22}$$

Considering $v(L) = 1 - \varepsilon_+$, $0 < \varepsilon_+ << 1$, we have $v(L) \longrightarrow 1$ when $L \to +\infty$.

By substituting (21) into (20), we conclude that the coefficients w_1, w_2 and w_3 satisfy

$$\begin{cases} w_1'(t) - K_\tau w_1(t) = 0, \\ w_2'(t) - K_\tau w_2(t) = -A_2w_1(t)^2, \\ w_3'(t) - K_\tau w_3(t) = -A_3w_1(t)^3 - 2A_2w_1(t)w_2(t). \end{cases} \tag{23}$$

The first equation of system (23) is linear and autonomous. It has the following characteristic equation:

$$\lambda + 2 + A_1 - 2cosh(\lambda\tau) = 0. \tag{24}$$

Knowing that $A_1 = -f'(1) > 0$, equation (24) has exactly two roots, one positive and one negative λ_- (figures 3 or 2). Since $w_1(t) = Ce^{\lambda t}$, where C is a constant, we choose the negative one to guarantee that $w_1(t) \longrightarrow 0$ as $t \to +\infty$.

Solving the system (23) and taking into account the conditions (22), when $t \ge L$ the asymptotic expansion takes the form

$$\begin{aligned} v(t) &= 1 - \varepsilon_+ e^{\lambda_-(t-L)} - \varepsilon^2 B_1(e^{2\lambda_-(t-L)} - e^{\lambda_-(t-L)}) \\ &\quad -\varepsilon_+^3(B_2e^{2\lambda_-(t-L)} + B_3e^{3\lambda_-(t-L)} - (B_2+B_3)e^{\lambda_-(t-L)}) \\ &\quad + O(\varepsilon_+^4). \end{aligned} \tag{25}$$

Notice that $\varepsilon_+ = 1 - v(L)$, B_1, B_2, B_3 and λ_- are constants depending on the Taylor expansion of f and the characteristic equation, when $t \ge L$. The expressions of B_1, B_2, B_3 are, respectively,

$$\begin{aligned} B_1 &= \tfrac{-A_2}{2\lambda_- + A_1 + 2 - 2cosh(\lambda_-\tau)}, \\ B_2 &= -2B_1^2, \\ B_3 &= \tfrac{-2A_1B_1 - A_3}{3\lambda_- + A_1 + 2 - 2cosh(3\lambda_-\tau)}. \end{aligned} \tag{26}$$

2.3 Test Problem

To analyze the convergence of the numerical scheme we take into account some test problems with known solutions.

There is one test problem with known solution which belongs to the class of the problems under study (3)-(4) and can be solved exactly. Let θ be a known positive constant and $f_\theta(v)$ given by

$$f_\theta(v) = \tfrac{1 + 2\theta(2v-1) - (1+\theta)(2v-1)^2 - \theta(3-2v)(2v-1)^3}{2(1-\theta(2v-1)^2)}, \tag{27}$$

with $-\infty < t < +\infty$. Then the solution of (4)-(3) is

$$v(t) = \frac{1 + tanh(t)}{2}, \tag{28}$$

where

$$\tau = tanh^{-1}(\sqrt{\theta}). \tag{29}$$

The solution of the test problem (27) can be used as an initial approximation for the numerical solution of the target problem (4) using the continuation method. We solve a sequence of equations of the form (4) where, in the right-hand side of the equation, f is replaced by f_θ. In each subsequent equation, f is replaced by f_α defined by

$$f_\alpha(v) = \alpha f_\theta(v) + (1-\alpha) f_{target}(v), \ \ 0 \le \alpha \le 1,$$

with f_{target} is the function f, defined by (5).

When $\alpha = 1$ we get the exact solution of the test problem. By the other hand, when $\alpha = 0$, we get the approximate solution of the target problem. We compute numerical solutions for the problem starting with an initial approximation taking $\alpha = 1$, and sequentially, α is decreased from 1 until 0. For each problem (each α), the correspondent initial approximation is the one obtained in the previous problem, with the previous value of α.

2.4 Method of steps

The main goal of this section is to extend the method of steps to the nonlinear case, using the results presented in [27]. It uses the ideas based on Bellman's method of steps for solving delay differential equations and some work introduced in [11, 8] where the method of steps is applied to an autonomous linear forward-backward differential equation. In the linear case, one solves the equation over successive intervals of unitary length. In the case of equation (4), we present the same idea for a non linear equation, considering sucessive intervals of length τ

$$v(t+\tau) = v'(t) - v(t-\tau) + g(v(t)), \quad t \in \mathbb{R}, \tag{30}$$

where $g(u) = 2u - f(u)$.

We can use formula (30) to construct a solution for equation (3) on an interval $[a, a+k\tau]$ (where k is an integer), $a \in \mathbb{R}$, starting from its initial values on $[a-2\tau, a]$; these starting values can be obtained from the asymptotic expansion (18).

Supposing that all necessary derivatives of f and v exist in $(a-2\tau, a]$, we may obtain the following expressions for the solution in the first two intervals $(a, a+\tau]$ and $(a+\tau, a+2\tau]$:

$$\begin{aligned} v(t+\tau) &= v'(t) - v(t-\tau) + g(v(t)), \qquad t \in (a, a+\tau], \\ v(t+2\tau) &= v'(t+\tau) - v(t) + g(v(t+\tau)), \\ &= v''(t) - v'(t-\tau) + g(v(t+\tau)) \\ &\quad + g'_v(v(t))v'(t) - v(t), \qquad t \in (a+\tau, a+2\tau]. \end{aligned} \tag{31}$$

If in first formula of (31) we set $g(v) = 2v$ (which corresponds to $f(v(t)) \equiv 0$) and $\tau = 1$ then we obtain the results of theorem 1 in [27], with $a(t) \equiv 1$, $b(t) \equiv -1$, $c(t) \equiv 2$.

Continuing this process, we can extend the solution to any interval, provided that the initial functions in the first two intervals with lenght τ are smooth enough functions and satisfy some simple relationships.

3 Newton's Method

When we consider a nonlinear problem, we often transform it into a sequence of linear problems. This process can be done using iterative schemes like the Newton's method.

When applying the Newton's method to a functional equation, it is usual to apply the concept of Frchet derivative of a certain nonlinear operator.

Definition 31*Let E_1 and E_2 be two normed spaces and consider an operator $P : V \subseteq E_1 \to E_2$, V open set. P is Frchet differentiable at $v \in V$ if exists an operator $F \in L(E_1, E_2)$ such as*

$$\lim_{\|H\|_{E_1} \to 0} \frac{\|P(v+h) - Pv - Fh\|_{E_2}}{\|H\|_{E_1}} = 0, \tag{32}$$

where $L(E_1, E_2)$ represents the set of all linear and continuous operators which apply E_1 in E_2 and $\|\cdot\|_E$ the norm in space E.

When operator F exists, it represents the Frchet derivative and $F = P'_v$.

So,

$$\begin{aligned} &P' : V \to L(E_1, E_2), \\ &v \to P'_v : E_1 \to E_2. \end{aligned} \tag{33}$$

When we consider a Frchet-differentiable operator P, with invertible derivative in a neighborhood of the solution $v(x)$, the equation $Pv = 0$ is equivalent to

$$v = v - (P'_{v_0})^{-1}(Pv), \quad v_0 \in E. \tag{34}$$

The Newton's method constructs a sequence of approximations using the iterative formula

$$v_{i+1} = v_i - (P'_{v=v_i})^{-1}(Pv_i), \quad v_0 \in E. \tag{35}$$

The equation (35) is equivalent to

$$P'_{v|v=v_i}(v_{i+1} - v_i) = -(Pv_i). \tag{36}$$

In the next section we will apply Newton's method to the nonlinear MTFDE (4). Equation (36) can be written as

$$P(v(t)) = 0, \tag{37}$$

where

$$P(v(t)) = v'(t) - v(t-\tau) - v(t+\tau) + 2v(t) - f(v(t)), \tag{38}$$

or

$$P(v(t)) = v'(t) - f(v(t)) - L^*(v(t)), \tag{39}$$

with

$$L^*v(t) = v(t-\tau) + v(t+\tau) - 2v(t). \tag{40}$$

Applying the Frchet derivative to operator P defined by (38) we have

$$P'(v(t)).u(t) = u'(t) - f'_v.u(t) - L^*(u(t)). \tag{41}$$

Writing the formula (36) in the case where P is defined by (39) and P' is defined by (41) with Newton's method applied to the equation (37) we obtain the following formula for Newton iterative process:

$$\begin{aligned} &(v'_{i+1}(t) - v'_i(t)) - f'_{v_i}.(v_{i+1}(t) - v_i(t)) - L^*(v_{i+1}(t) - v_i(t)) \\ &= -(v'_i(t) - f(v_i(t)) - L^*(v_i(t))). \end{aligned} \tag{42}$$

Equation (42) can also be written as

$$v'_{i+1}(t) - f'_{v_i}.(v_{i+1}(t) - v_i(t)) - L^*(v_{i+1}(t)) = f(v_i(t)). \tag{43}$$

Or, in a more compact form,

$$P'_{v_i}.v_{i+1}(t) = f(v_i(t)) - f'_{v_i}v_i(t). \tag{44}$$

4 Numerical Scheme

The asymptotic analysis of the solutions (as $t \to \pm\infty$) allows us to transfer the boundary conditions and to reduce the present problem to an equivalent problem on a finite interval $[-L-\tau, L+\tau]$, where L is a sufficiently large number. Instead of model (3)-(4) we propose a boundary value problem on $[-L, L]$, with the boundary conditions given at $[-L-\tau, -L]$ and $[L, L+\tau]$, for some positive large enough integer L. Using the asymptotic properties of the solutions, for large values of $|t|$, formulae (18) and (25), we obtain approximations of the solution on certain intervals $[-L-\tau, -L]$ and $[L, L+\tau]$.

Hence, instead of solving equation (4) on the entire real line, we will solve it for $t \in [-L, L]$; in this case boundary conditions (3) are replaced by

$$\begin{cases} v(t) = \phi_0(t), & t \in [-L-\tau, -L], \\ v(t) = \phi_1(t), & t \in [L, L+\tau], \end{cases} \tag{45}$$

where ϕ_0 and ϕ_1 correspond to the truncated forms of the asymptotic formulae given by (18) and (25) respectively.

Moreover, we recall that the needed solution must satisfy the condition (6).

We shall now describe a numerical scheme to solve problem (3)-(4) with condition (6), where this problem is first reduced to the form (4) with conditions (6) and (45).

A feature of proposed algorithm, which makes it substantially different from the methods developed previously in [1, 6] is that it is built in two stages:

1. Compute the shift τ and define the asymptotic behavior of the solution as $t < -L$ and $t > L$.
2. With the estimate of τ we can solve numerically the problem (4) with conditions (6) and (45) using the Newton's method, described in 3, and solving each linear iterate by the collocation method.

1. **First stage**:

   (a) In order to compute the solution on interval $[-L-\tau, -L]$, we must fix a certain initial value for τ and solve the characteristic equation (17). Note that we impose $L = K\tau$, where K is an integer.

   (b) Knowing the characteristic values λ_+ and λ_- we compute the approximate solution at $[-L-\tau, -L]$, using formulae (18) and (19), and at $[L, L+\tau]$, using the asymptotic expansion (25) and (26). Moreover, we can compute the first K derivatives of solution (where $K = \frac{L}{\tau}$), which will be needed to apply the method of steps. In order to compute these values we must fix certain initial guesses for ε and ε_+.

   (c) Using the estimated values of v and its first K derivatives at $-L-\tau$ and $-L$, we compute the solution values $v(-L+\tau)$, $v(-L+2\tau)$, $\ldots$, $v(0)$, recursively, using formula (30). Then, we compute ε from the condition (6).

   (d) In the same way, we compute the values of $v(L-\tau)$, $v(L-2\tau)$, $\ldots$, $v(0)$, starting the values of v and its K derivatives at $v(L+\tau)$ and $v(L)$. In this case, we have to apply the formula (30) backwards. The correct value of ε_+ is determined from the condition that (6).

   (e) Finally, we have to compute the value of τ, from the condition

   $$\lim_{t \to o^+} v'(t) = \lim_{t \to o^-} v'(t) \tag{46}$$

   (differentiability of the solution at $t = 0$). The left and the right-hand sides limits of v', at $t = 0$, are computed using again formula (30), from $-L$ to 0 and from L to 0, respectively. More precisely, differentiating both sides of (30), we obtain a recursive formula that can be used to compute the derivative of v.

   (f) At the end of this process, we know that the values of τ, ε and ε_+, as well as the values of the solution and its first derivative at $t = -L$, $t = -L+\tau, \ldots$.

2. **Second stage:**

   (a) Here we must solve the problem (4), (45), for a known value of τ, using the Newton's method. With this purpose, we must solve a sequence of linear equations of the form (42), using the collocation method derived in [13, 29]. In each iterate (43) we search a solution which satisfies boundary conditions (45). Note that equation (43) correspond to the non-autonomous linear equation considered in [13].

   (b) When computing the first iterative $v_1(t)$, for the test problem (where f is given by (27)), we take as v_0 the function (28).

   (c) Then, we compute a sequence of iterates $v_1(t)$, $v_2(t), \ldots, v_n(t)$, until the following condition is satisfied

   $$\|v_n(t) - v_{n-1}(t)\| \leq tol, \tag{47}$$

   where tol is a small enough positive constant.

   (d) Since the exact solution for the test problem is known we can compare the numerical results with the exact ones.

5 Numerical Results

In this section, we present and discuss some numerical results which illustrate the performance of the numerical scheme proposed in the previous section, for the solution of (3)-(4).

First, the algorithm is tested using a problem with known solution, the test problem (27), where we take $f = f_\theta$. We analyze both the convergence of the method and the accuracy of the obtained results.

5.1 Test Problem

In Table 1, we present different iterates of the numerical solution for the test problem described in section 2.3, taking $\theta = 0.35$, for $t = -1.2$ and $t = -0.2$. Once the exact solution has $\tau = 0.680136$, the initial value taken for τ was $\tau = 0.680$. The initial guess, in iterative Newton process, is given by $v(ct)$, where is computed by (28) or (29) and c is a certain constant.

Table 1: Test Problem: Sequential solution iterates by Newton's method for test problem. Tolerance: $tol = 10^{-6}$. $\theta = 0.35$, $t = -1.2$, $t = -0.2$.

	$t=-1.2$		$t=-0.2$	
	Solution	Absolute error	Solution	Absolute error
exact	**0.083172696**		**0.401312340**	
iter. 1	0.075474	7.70×10^{-3}	0.456799	5.55×10^{-2}
iter. 2	0.0831057	6.69×10^{-5}	0.410576	9.26×10^{-3}
iter. 3	0.0831710	1.70×10^{-6}	0.402767	1.45×10^{-3}
iter. 4	0.0831712	1.49×10^{-6}	0.401300	1.23×10^{-5}
iter. 5	0.0831714	1.29×10^{-6}	0.401310	2.34×10^{-6}
iter. 6	0.0831715	1.19×10^{-6}	0.401310	2.33×10^{-6}
iter. 7	0.0831726	9.65×10^{-8}	0.401312	3.40×10^{-7}

The iterative process seems to be convergent when compared with the exact value of solution. The estimated value of $\tau = 0.680136$ has an absolute error of 2.7037×10^{-7}.

The values of v_i at $t = -L + i\tau$, $i = 1, 2, \ldots, 7$ and $K = 3$ are given in Table 2. Again, the computed values have reasonable accuracy.

Table 2: Test Problem: Solution estimates v_i and absolute error ε_i at $t = -L + i\tau$, $i = 1, 2, \ldots, 7$, $K = 3$, $\theta = 0.35$.

$\mathbf{i}$	$\mathbf{v_i = v(-L + i\tau)}$	$\varepsilon_\mathbf{i}$
1	$1.66131563 \times 10^{-2}$	1.3644×10^{-7}
2	$6.17718700 \times 10^{-2}$	6.0553×10^{-8}
3	$2.04195990 \times 10^{-1}$	1.0833×10^{-7}
4	0.5	0
5	$7.95803735 \times 10^{-1}$	1.6635×10^{-7}
6	$9.38228209 \times 10^{-1}$	1.3958×10^{-7}
7	$9.83387041 \times 10^{-1}$	6.0923×10^{-8}

In Table 3 we present the estimates of convergence order $p = log_2 \varepsilon_{2N} / log_2 \varepsilon_N$ and the absolute error of the numerical solution ε_N, considering the test problem with $\theta = 0.35$.

Table 3: Test Problem: $\theta = 0.35$, $K = 2$ and $K = 3$. N: number of subintervals. Convergence order estimate p and absolute error ε (norm-2).

	$K=2,\ L\approx 2.04$		$K=3,\ L\approx 2.72$	
N	ε	p	ε	p
8	$1.52\times^{-8}$		$2.83\times^{-7}$	
16	$3.48\times^{-9}$	2.13	$1.08\times^{-7}$	1.40
32	$3.25\times^{-10}$	2.07	$3.05\times^{-8}$	1.82
64	$1.99\times^{-10}$	2.05	$7.95\times^{-9}$	1.94
128	$4.89\times^{-11}$	2.04	$2.02\times^{-9}$	1.98

The results are accurate and the estimates of convergence order p, using 2-norm, agree with the expected value, $p = 2$, taking into account the analysis of the linear case (see Section 2 in [13]). The absolute error of the numerical solution is about 4.89×10^{-11} and 2.02×10^{-9} when L is close to 2 and 2.7 respectively ($K = 2, 3$) and $N = 128$.

5.2 Target Problem

Table 4 presents the number of Newton iterates until the process stops for each value of α in the continuation method, taking the tolerance parameter $tol = 10^{-6}$. Taking into account the number of iterates, the process seems to behave correctly in the sense of convergence.

Table 4: Target Problem: $a = 0.05$, $b = 15$, $N = 51$, $N = 81$. $\alpha = 0, 0.2, 0.4, \ldots, 1$. Number of iterations n_i of the Newton method.

α	$\mathbf{N = 50}$	$\mathbf{N = 8}$
0	8	8
0.2	9	8
0.4	10	9
0.6	9	7
0.8	8	8
1.0	7	6

Figure 4 represents the graphics of numerical solutions on $[-2, 2]$ obtained by continuation, for $\alpha = 0, 0.2, 0.4, \ldots, 1$. The case $\alpha = 0$ corresponds to the numerical solution of the target problem, $\alpha = 1$ corresponds to the numerical solution of the test problem. In the test problem we use $\theta = 0.35$. The target problem was considered with f_{target} given by (5), $a = 0.05$, $b = 15$. When α decreases from one to zero, the curves became more stiff. Also, at the neighborhood of $t = 0$, the slope becomes greater and the graphics approaches to vertical. Notice that in all graphics we have $v(0) = 0.5$.

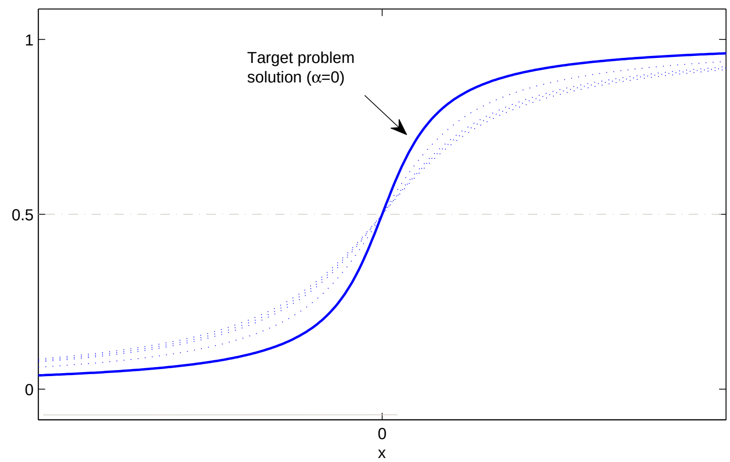

Fig. 4: Numerical approximation f_α using approximation method. $N = 81$, $a = 0.05$, $b = 15$, $\alpha = 0, 0.2, 0.4, \ldots, 1$.

6 Conclusions

We propose a scheme to solve numerically a nonlinear mixed type functional differential equation defined in an unbounded domain. Such equation models the nerve conduction in myelinated axons. The convergence of the numerical scheme is verified solving numerically some test problems with known solutions. Numerical results displayed in Table 3 are in accordance with the results presented in [6] and in [15] where was proposed a numerical scheme which does not consider the continuation method. The accuracy of such results is better. The numerical solution of the target problem converged as we can see in Figure 4. Also, we can confirm the expected behavior of the solution: the solution at any node should be monotone increasing. This arises from the current-voltage relation: once a node is turned on, it cannot return to the rest potential $v = 0$.

A question in study is how the solution of equation (4) will be affected by changing the parameters of the problem. Some simulations were done, but the results and discussion are not presented in this article. Actually, some numerical results using finite element method instead collocation methods were obtained for test problem; these last computations give more accurate results than the results displayed in Table 3. We are also implementing a new numerical scheme based on homotopy analysis method (HAM) following the work in [20], where some differential delay equations are studied. In [22], it is considered a preliminary approach to a non linear mixed type functional equation from acoustics using HAM. Another idea is the construction of a numerical scheme where a different set of basis functions is considered, for example, the radial basis functions (see [4, 5, 30]), similarly the approach done in [26, 21].

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