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Ostrowski and Simpson Type Inequalities for Strongly Reciprocally Convex Functions

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Abstract: In this paper, we establish some new Ostrowski and Simpson type inequalities for the class of strongly reciprocally convex functions. These bounds are very useful in applications.

Keywords: Harmonically convex functions, Ostrowski type inequalities, Simpson type inequalities

1 Introduction

Inequalities continue to play an essential role in mathematics. The subject is perhaps the last field that is comprehended and used by mathematicians that work in all the areas of that noble science. Since the seminal work *inequalities* [13] *of Hardy, Littlewood and Polya* mathematicians have labored to extend and sharpen the earlier classical inequalities. New inequalities are discovered every year, some for their intrinsic interest whilst others flow from results obtained in various branches of mathematics. The study of inequalities reflects the many and various aspects of mathematics. There is, on the one hand, the systematic search for the basic principles and the study of inequalities for their own sake. On the other hand, there are many applications in a wide variety of fields from mathematical physics to biology and economics. In this article we will present the counterparts for *strongly reciprocally convex functions* of two well-known inequalities: those known as Ostrowski and Simpson (inequalities).

In 1938, Alexander Markovich Ostrowski (see [26]) proved the following integral inequality (1). Ostrowski considered the problem of estimating the deviation of a function from its integral mean. To be precise, for any continuous function f on $[a,b] \subseteq \mathbb{R}$ which is differentiable on (a,b) and with the property that $|f'(x)| \le M$ for all $x \in (a,b)$, the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)M, \quad (1)$$

holds for every $x \in (a,b)$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant. The inequality (1) is well known in the literature as the Ostrowski inequality. Recently, several generalizations of the Ostrowski inequality for mappings of bounded variation and for Lipschitzian, monotonic, absolutely continuous and n-times differentiable mappings, with error estimates for some special means and for some numerical quadrature rules, have been considered by many authors.

Many researchers have given considerable attention to the inequality (1) and several generalizations, extensions and related results have appeared in the literature. For some results which generalize, improve, and extend the above inequality, see [9,1,3,5,10,11,23].

One of the two goal of this paper is to establish an Ostrowski type inequality for strongly reciprocally convex functions.

One of the fundamental results in numerical integration is the Simpson's inequality which states:

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Theorem 1([34]). *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) with $\|f^{(4)}\|_\infty := \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\|f^{(4)}\|_\infty(b-a)^4. \tag{2}$$

This inequality gives an error bound for the classical Simpson quadrature formula, which is one of the most used quadrature formulae in practical applications. It is well known that if the function f is neither four times differentiable nor its fourth derivative is bounded on (a,b), then we cannot apply the classical Simpson quadrature formula.

For recent results and generalizations concerning Simpson's inequality see [2,7,8,11,6,12,15,20,21,31,32,36,35,27,28] and the references therein.

Another goal of this paper is to establish a Simpson-type inequality for strongly reciprocally convex functions.

The setup of this paper is as follows. In section 2 we review the necessary background on strongly convex functions and on strongly reciprocally convex function. Next, in section 3 we prove our main results.

2 Preliminaries

Convexity is one of the most natural, fundamental, and important notions in mathematics whose applications go down to the times of Archimedes (Circa 287 B.D.). Convex functions were introduced by J. L. W. V. Jensen over 100 years ago and since then they have been a subject of intensive investigations. In recent years several extensions and generalizations have been given for this classical notion.

A function $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ is said to be convex if whenever $x,y\in[a,b]$ and $t\in[0,1]$, the following inequality holds:

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y).$$

A significant generalization of the notion of convex functions is that of Strongly convex function introduced by B. Polyak in [29].

Definition 1(See [14,22,29,30]). *Let I be a interval of $\mathbb{R}$ and let $c>0$. A function $f:I\to\mathbb{R}$ is called strongly convex with modulus c if*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2 \tag{3}$$

for all $x,y\in I$ and $t\in[0,1]$.

The usual notion of convex function correspond to the case $c=0$. The followings facts are well known (see e.g. [30]): If f is strongly convex, then it is bounded from below, its level sets $\{x\in I: f(x)\leq\lambda\}$ are bounded for each λ and f has a unique minimum on every closed subinterval of I ([24, p. 268]). Any strongly convex function defined on a real interval admits a quadratic support at every interior point of its domain.

Strongly convex functions play an important role in optimization theory and mathematical economics. The notion of strongly convex function is of great use in optimization problems, as it can significantly increase the rate of convergence of first-order methods such as projected subgradient descent [18], or more generally the forward-backward algorithm [5, Example 27.12].

The following result states the relation between convex and strongly convex functions.

Theorem 2([25]). *Let D be a convex subset of $\mathbb{R}$ and let c be a positive constant. A function $f:D\to\mathbb{R}$ is strongly convex with modulus c if and only if the function $g(x)=f(x)-cx^2$ is convex.*

In [33], E. Set et al. proved the following result on Ostrowski's type inequality for strongly-convex functions.

Theorem 3. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|g'|$ is strongly-convex on $[a,b]$ with respect to $c>0$, $|f'|\leq M$ and $M\geq\max\left\{\frac{c(x-a)^2}{6},\frac{c(b-x)^2}{6}\right\}$, then the following inequality holds;*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(x-a)^2}{2(b-a)}\left(M-\frac{c(x-a)^2}{6}\right)+\frac{(b-x)^2}{2(b-a)}\left(M-\frac{c(b-x)^2}{6}\right),$$

for all $x,y\in[a,b]$ and $t\in(0,1)$.

Given to numbers x,y in an interval $I\subset\mathbb{R}\backslash\{0\}$ the quantity

$$\frac{xy}{tx+(1-t)y}$$

is known as the harmonic mean of x and y. Recently, in [17], Işcan introduced the notion of harmonically convex function, which is defined as follows.

Definition 2. *Let $I\subseteq\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq tf(y)+(1-t)f(x)$$

for all $x,y\in I$ and $t\in(0,1)$.

In [4], we introduced the notion of strongly reciprocally convex function:

Definition 3. *Let I be an interval in $\mathbb{R}\setminus\{0\}$ and let $c\in(0,\infty)$. A function $f:I\to\mathbb{R}$ is said to be* strongly reciprocally convex *with modulus c on I, if the inequality*

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq tf(y)+(1-t)f(x)-ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2 \tag{4}$$

holds, for all $x,y\in I$ and $t\in[0,1]$.

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The symbol $\mathrm{SRC}_{(I,c)}$ will denote the class of functions that satisfy the inequality (4). We obtained the following properties for this kind of functions:

Theorem 4. *Let $f,g : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be two functions and let $c_1,c_2,k \in \mathbb{R}_+$. Then*

1. *If $f \in SRC_{(I,c_1)}$ and $g \in SRC_{(I,c_2)}$, then $f+g \in SRC_{(I,c_1+c_2)} \cap SRC_{(I,c_1)} \cap SRC_{(I,c_2)}$,*
2. *If $f \in SRC_{(I,c)}$, then $kf \in SRC_{(I,kc)}$. In addition, if $k \geq 1$, then $kf \in SRC_{(I,c)}$.*

Moreover, in [4], we also obtained the following complete characterization:

Theorem 5. *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval and $c \in (0,\infty)$.*

1. *If $f \in SRC_{(I,c)}$, then f es harmonically convex.*
2. *$f \in SRC_{(I,c)}$ if and only if the function $g : I \to \mathbb{R}$, defined by $g(x) := f(x) - \dfrac{c}{x^2}$ es harmonically convex.*
3. *If $f \in SRC_{(I,c)}$, then the function $\varphi = f + \varepsilon \in SRC_{(I,c)}$, for any constants ε. If $f : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ and if we consider the function $g : \left[\dfrac{1}{b},\dfrac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) := f\left(\dfrac{1}{t}\right)$, then $f \in SRC_{([a,b],c)}$ if and only if g is strongly convex in $\left[\dfrac{1}{b},\dfrac{1}{a}\right]$.*

3 Main results

In this section, we derive our main results. We establish some inequalities of Ostrowski and Simpson type for strongly reciprocally convex functions.

Ostrowski type inequalities

For the reader's convenience, we recall here the definitions of hypergeometric functions that are employed in the following discussion.

Definition 4([19]). *For the real or complex numbers a,b,c other than $0,-1,-2,\ldots$, the hypergeometric series of a,b and c is defined by*

$$
\begin{aligned}
{}_2F_1(a,b;c;z) &= 1 + \frac{a\cdot b}{c}\cdot\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\cdot\frac{z^2}{2!} + \cdots \\
&= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m}\cdot\frac{z^m}{m!}.
\end{aligned}
$$

Here, $(\alpha)_m := \begin{cases} 1 & m=0 \\ \alpha(\alpha+1)\cdots(\alpha+m-1), & m>0, \end{cases}$ which has the integral form:

$$
{}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,
$$

where $|z|<1$, $c>b>0$ and $\beta(x,y) := \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}dt.$

We will need the following lemma, whose proof can be found in [18].

Lemma 1([18]). *Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a,b \in I$, with $a<b$. If $f' \in L[a,b]$, then*

$$
\begin{aligned}
& f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \\
&= \frac{ab}{b-a}\left\{(x-a)^2\int_0^1 \frac{t}{(ta+(1-t)x)^2} f'\left(\frac{ax}{at+(1-t)x}\right)dt\right. \\
&\quad \left. -(b-x)^2\int_0^1 \frac{t}{(tb+(1-t)x)^2} f'\left(\frac{bx}{bt+(1-t)x}\right)dt\right\}.
\end{aligned}
$$

Theorem 6. *Let $I \subset (0,\infty)$ be an real interval and $f : I \to \mathbb{R}$ be a differentiable function on I°. Suppose that $a,b \in I^\circ$ with $a<b$, and that $f' \in L[a,b]$. If $|f'|^q \in SRC_{([a,b],c)}$ for $q \geq 1$, then for all $x \in [a,b]$, we have*

$$
\begin{aligned}
& \left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \right| \\
&\leq \frac{ab}{b-a}\left\{(x-a)^2\left[\lambda_1(a,x,q,q)|f'(x)|^q + \lambda_2(a,x,q,q)|f'(a)|^q\right.\right. \\
&\qquad \left. -c\left(\frac{1}{a}-\frac{1}{x}\right)^2\lambda_3(a,x,q,q)\right]^{\frac{1}{q}} \\
&\quad +(b-x)^2\left[\lambda_4(b,x,q,q)|f'(x)|^q + \lambda_5(b,x,q,q)|f'(b)|^q\right. \\
&\qquad \left.\left. -c\left(\frac{1}{x}-\frac{1}{b}\right)^2\lambda_6(a,x,q,q)\right]^{\frac{1}{q}}\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_1(a,x,\nu,\rho) &:= \frac{\beta(\rho+2,1)}{x^{2\nu}}\cdot {}_2F_1\left(2\nu,\rho+2;\rho+3;1-\frac{a}{x}\right), \\
\lambda_2(a,x,\nu,\rho) &:= \frac{\beta(\rho+1,2)}{x^{2\nu}}\cdot {}_2F_1\left(2\nu,\rho+1;\rho+3;1-\frac{a}{x}\right), \\
\lambda_3(a,x,\nu,\rho) &:= \frac{\beta(\rho+2,2)}{x^{2\nu}}\cdot {}_2F_1\left(2\nu,2;\rho+4;1-\frac{a}{x}\right), \\
\lambda_4(b,x,\nu,\rho) &:= \frac{\beta(1,\rho+2)}{b^{2\nu}}\cdot {}_2F_1\left(2\nu,1;\rho+3;1-\frac{x}{b}\right), \\
\lambda_5(b,x,\nu,\rho) &:= \frac{\beta(2,\rho+1)}{b^{2\nu}}\cdot {}_2F_1\left(2\nu,2;\rho+3;1-\frac{x}{b}\right), \\
\lambda_6(b,x,\nu,\rho) &:= \frac{\beta(2,\rho+2)}{b^{2\nu}}\cdot {}_2F_1\left(2\nu,2;\rho+4;1-\frac{x}{b}\right),
\end{aligned}
$$

Proof. From Lemma 1, for all $x \in [a,b]$, we have

$$
\begin{aligned}
& \left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2} \right| \\
&= \left| \frac{ab}{b-a}\left\{(x-a)^2\int_0^1 \frac{t}{(ta+(1-t)x)^2} f'\left(\frac{ax}{at+(1-t)x}\right)dt\right.\right. \\
&\quad \left.\left. -(b-x)^2\int_0^1 \frac{t}{(tb+(1-t)x)^2} f'\left(\frac{bx}{bt+(1-t)x}\right)dt\right\}\right|,
\end{aligned}
$$

applying Hölder inequality to the integral on the right side of the above inequality, we obtain

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$$\begin{aligned}
&\leq \frac{ab(x-a)^2}{b-a}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\left[\int_0^1\left(\frac{t}{(ta+(1-t)x)^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|\right)^q dt\right]^{\frac{1}{q}} \\
&+\frac{ab(b-x)^2}{b-a}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\left[\int_0^1\left(\frac{t}{(tb+(1-t)x)^2}\left|f'\left(\frac{bx}{tb+(1-t)x}\right)\right|\right)^q dt\right]^{\frac{1}{q}} \\
&= \frac{ab(x-a)^2}{b-a}\left[\int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|^q dt\right]^{\frac{1}{q}} \\
&+\frac{ab(b-x)^2}{b-a}\left[\int_0^1 \frac{t^q}{(tb+(1-t)x)^{2q}}\left|f'\left(\frac{bx}{tb+(1-t)x}\right)\right|^q dt\right]^{\frac{1}{q}} \\
&\leq \frac{ab(x-a)^2}{b-a}\left[\int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}}[t|f'(x)|^q+(1-t)|f'(a)|^q\right. \\
&\qquad \left.-ct(1-t)\left(\frac{1}{a}-\frac{1}{x}\right)^2\right]dt\Bigg]^{\frac{1}{q}} \\
&+\frac{ab(b-x)^2}{b-a}\left[\int_0^1 \frac{t^q}{(tb+(1-t)x)^{2q}}[t|f'(x)|^q+(1-t)|f'(b)|^q\right. \\
&\qquad \left.-ct(1-t)\left(\frac{1}{b}-\frac{1}{x}\right)^2\right]^{\frac{1}{q}}\Bigg]. \qquad (5)
\end{aligned}$$

Thus, by simple calculations, we have:

$$\begin{aligned}
\lambda_1(a,x,q,q) &= \int_0^1 \frac{t^{q+1}}{(ta+(1-t))^{2q}}dt \\
&= \frac{\beta(q+2,1)}{x^{2q}}\,.{}_2F_1\left(2q,q+2;q+3;1-\frac{a}{x}\right), \\
\lambda_2(a,x,q,q) &= \int_0^1 \frac{t^q(1-t)}{(ta+(1-t))^{2q}}dt \\
&= \frac{\beta(q+1,2)}{x^{2q}}\,.{}_2F_1\left(2q,q+1;q+3;1-\frac{a}{x}\right), \\
\lambda_3(a,x,q,q) &= \int_0^1 \frac{t^q t(1-t)}{(ta+(1-t))^{2q}}dt \\
&= \frac{\beta(q+2,2)}{x^{2q}}\,.{}_2F_1\left(2q,q+2;q+4;1-\frac{a}{x}\right), \\
\lambda_4(b,x,q,q) &= \int_0^1 \frac{t^{q+1}}{(tb+(1-t))^{2q}}dt \\
&= \frac{\beta(1,q+2)}{b^{2q}}\,.{}_2F_1\left(2q,1;q+3;1-\frac{x}{b}\right), \\
\lambda_5(b,x,q,q) &= \int_0^1 \frac{t^q(1-t)}{(tb+(1-t))^{2q}}dt \\
&= \frac{\beta(2,q+1)}{b^{2q}}\,.{}_2F_1\left(2q,2;q+3;1-\frac{x}{b}\right), \\
\lambda_6(b,x,q,q) &= \int_0^1 \frac{t^q t(1-t)}{(tb+(1-t))^{2q}}dt \\
&= \frac{\beta(2,q+2)}{b^{2q}}\,.{}_2F_1\left(2q,2;q+4;1-\frac{x}{b}\right).
\end{aligned}$$

Substituting these values into the inequality (5), we obtain the desired result. This completes the proof.

Simpson type inequality

The following facts have been greatly motivated by the important work of Işcan [16]. The purpose of this subsection is to give an inequality of Simpson type for strongly reciprocally convex functions.

In order to prove our main results we need the following lemma.

Lemma 2. *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a,b \in I$ with $a<b$. If $f' \in L[a,b]$ then for $\lambda \in [0,1]$:*

$$\begin{aligned}
&(1-\lambda)f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\
&= \frac{ab(b-a)}{2}\left[\int_0^{\frac{1}{2}} \frac{\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt+\int_{\frac{1}{2}}^1 \frac{2-\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt\right], \qquad (6)
\end{aligned}$$

where $A_t = tb+(1-t)a$.

The proof of this lemma can be found in [16].

Now using Lemma 2 we prove the second main result of this paper.

Theorem 7. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a<b$, and suppose that $f' \in L[a,b]$. If $|f'|^q \in SRC_{([a,b],c)}$ for $q \geq 1$ then the following inequality holds for $\lambda \in [0,1]$:*

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\left\{C_1^{1-\frac{1}{q}}(\lambda;a,b)\left[C_2(\lambda;a,b)|f'(a)|^q+C_3(\lambda;a,b)|f'(b)|^q\right.\right. \\
&\qquad \left.-c\left(\frac{1}{a}-\frac{1}{b}\right)^2 C_4(\lambda;a,b)\right]^{\frac{1}{q}} \\
&+C_1^{1-\frac{1}{q}}(\lambda;b,a)\left[C_3(\lambda;b,a)|f'(a)|^q+C_2(\lambda;b,a)|f'(b)|^q\right. \\
&\qquad \left.\left.-c\left(\frac{1}{a}-\frac{1}{b}\right)^2 C_4(\lambda;b,a)\right]^{\frac{1}{q}}\right\}, \qquad (7)
\end{aligned}$$

where

$$\begin{aligned}
C_1(\lambda;u,v) &:= \frac{1}{(v-u)^2}\left[-4+\frac{[\lambda(v-u)+2u](3u+v)}{u(u+v)}+2\ln\left(\frac{2u(u+v)}{[\lambda(v-u)+2u]^2}\right)\right], \\
C_2(\lambda;u,v) &:= \frac{1}{(v-u)^3}\left[[\lambda(v-u)+4u]\ln\left(\frac{[\lambda(v-u)+2u]^2}{2u(u+v)}\right)\right. \\
&\qquad \left.-\frac{[\lambda(v-u)+2u](5u+3v)}{u+v}+7u+v\right], \\
C_3(\lambda;u,v) &:= C_1(\lambda;u,v)-C_2(\lambda;u,v), \\
C_4(\lambda;u,v) &:= \frac{1}{(b-a)^4}\left[\left\{\lambda(b^2-a^2)+2a(a+2b)\right]\ln\left(\frac{[\lambda(b-a)+2a]^2}{2a(a+b)}\right)\right. \\
&\qquad -\frac{[a^2+10ab+5b^2][\lambda(b-a)+2a]}{2(a+b)}+\frac{7a^2+30ab+3b^2}{4} \\
&\qquad \left.-\frac{[\lambda(b-a)+2a]^2}{2}\right\},
\end{aligned}$$

with $u,v>0$.

Proof. Let $A_t = bt+(1-t)a$, with $t \in [0,1]$. From Lemma 2 and using the Hölder inequality, for $\lambda \in [0,1]$ and $\frac{1}{p} :=$

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$1-\frac{1}{q}$, we get

$$
\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&=\left|\frac{ab(b-a)}{2}\left[\int_0^{\frac12}\frac{\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt+\int_{\frac12}^1\frac{2-\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt\right]\right| \\
&\le\frac{ab(b-a)}{2}\left[\int_0^{\frac12}\frac{|\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt+\int_{\frac12}^1\frac{|2-\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right] \\
&=\frac{ab(b-a)}{2}\left[\int_0^{\frac12}\left(\frac{|\lambda-2t|}{A_t^2}\right)^{\frac1p}\left(\frac{|\lambda-2t|}{A_t^2}\right)^{\frac1q}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right. \\
&\left.+\int_{\frac12}^1\left(\frac{|2-\lambda-2t|}{A_t^2}\right)^{\frac1p}\left(\frac{|2-\lambda-2t|}{A_t^2}\right)^{\frac1q}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right] \\
&\le\frac{ab(b-a)}{2} \\
&\left\{\left(\int_0^{\frac12}\left[\left(\frac{|\lambda-2t|}{A_t^2}\right)^{\frac1p}\right]^p dt\right)^{\frac1p}\left(\int_0^{\frac12}\left[\left(\frac{|\lambda-2t|}{A_t^2}\right)^{\frac1q}\left|f'\left(\frac{ab}{A_t}\right)\right|\right]^q dt\right)^{\frac1q}\right. \\
&+\left(\int_{\frac12}^1\left[\left(\frac{|2-\lambda-2t|}{A_t^2}\right)^{\frac1p}\right]^p dt\right)^{\frac1p} \\
&\left.\left(\int_{\frac12}^1\left[\left(\frac{|2-\lambda-2t|}{A_t^2}\right)^{\frac1q}\left|f'\left(\frac{ab}{A_t}\right)\right|\right]^q dt\right)^{\frac1q}\right\} \\
&=\frac{ab(b-a)}{2}\left\{\left(\int_0^{\frac12}\frac{|\lambda-2t|}{A_t^2}dt\right)^{1-\frac1q}\left(\int_0^{\frac12}\frac{|\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{\frac1q}\right. \\
&\left.+\left(\int_{\frac12}^1\frac{|2-\lambda-2t|}{A_t^2}dt\right)^{1-\frac1q}\left(\int_{\frac12}^1\frac{|2-\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{\frac1q}\right\} \\
&\le\frac{ab(b-a)}{2}\left\{\left(\int_0^{\frac12}\frac{|\lambda-2t|}{A_t^2}dt\right)^{1-\frac1q}\right. \\
&\left(\int_0^{\frac12}\frac{|\lambda-2t|}{A_t^2}\left[t|f'(a)|^q+(1-t)|f'(b)|^q-ct(1-t)\left(\frac1a-\frac1b\right)^2\right]dt\right)^{\frac1q} \\
&+\left(\int_{\frac12}^1\frac{|2-\lambda-2t|}{A_t^2}dt\right)^{1-\frac1q} \\
&\left.\times\left(\int_{\frac12}^1\frac{|2-\lambda-2t|}{A_t^2}\left[t|f'(a)|^q+(1-t)|f'(b)|^q-ct(1-t)\left(\frac1a-\frac1b\right)^2\right]dt\right)^{\frac1q}\right\}.
\end{aligned}
$$

(7) holds when we make the choice

$$
\begin{aligned}
C_1(\lambda;a,b)&=\int_0^{\frac12}\frac{|\lambda-2t|}{A_t^2}dt=\int_0^{\frac{\lambda}{2}}\frac{\lambda-2t}{[bt+(1-t)a]^2}dt \\
C_2(\lambda;a,b)&=\int_0^{\frac12}\frac{|\lambda-2t|t}{A_t^2}dt, \\
C_3(\lambda;a,b)&=\int_0^{\frac12}\frac{|\lambda-2t|(1-t)}{A_t^2}dt, \qquad \text{y} \\
C_4(\lambda;a,b)&=\int_0^{\frac12}\frac{|\lambda-2t|t(1-t)}{A_t^2}dt.
\end{aligned}
$$

Now it can easily be shown that

$$
\begin{aligned}
C_1(\lambda;a,b)&=\frac{1}{(b-a)^2}\left[-4+\frac{[\lambda(b-a)+2a](3a+b)}{a(a+b)}\right. \\
&\left.+2\ln\left(\frac{2a(a+b)}{[\lambda(b-a)+2a]^2}\right)\right], \\
C_2(\lambda;a,b)&=\frac{1}{(b-a)^3}\left\{[\lambda(b-a)+4a]\ln\left(\frac{[\lambda(b-a)+2a]^2}{2a(a+b)}\right)\right. \\
&\left.-\frac{[\lambda(b-a)+2a](5a+3b)}{a+b}+7a+b\right\}, \\
C_3(\lambda;a,b)&=C_1(\lambda;a,b)-C_2(\lambda;a,b).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
C_4(\lambda;a,b)&=\int_0^{\frac12}\frac{|\lambda-2t|t(1-t)}{A_t^2}dt=\int_0^{\frac12}\left[\frac{|\lambda-2t|t}{A_t^2}-\frac{|\lambda-2t|t^2}{A_t^2}\right]dt. \\
&=\frac{1}{(b-a)^4}\left[\left\{\lambda(b^2-a^2)+2a(a+2b)\right]\ln\left(\frac{[\lambda(b-a)+2a]^2}{2a(a+b)}\right)\right. \\
&-\frac{[a^2+10ab+5b^2][\lambda(b-a)+2a]}{2(a+b)}+\frac{7a^2+30ab+3b^2}{4} \\
&\left.-\frac{[\lambda(b-a)+2a]^2}{2}\right\}.
\end{aligned}
$$

This concludes the proof.

4 Some applications

Corollary 1. *With the same hypotheses and notations of Theorem 6, if, in addition, $|f'(x)|\le M$, $x\in[a,b]$, then for all $x\in[a,b]$,*

$$
\begin{aligned}
&\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \\
&\le\frac{ab}{b-a}\left\{(x-a)^2\left[\lambda_1(a,x,q,q)M^q+\lambda_2(a,x,q,q)M^q\right.\right. \\
&\left.-c\left(\frac1a-\frac1x\right)^2\lambda_3(a,x,q,q)\right]^{\frac1q} \\
&+(b-x)^2\left[\lambda_4(b,x,q,q)M^q+\lambda_5(b,x,q,q)M^q\right. \\
&\left.\left.-c\left(\frac1b-\frac1x\right)^2\lambda_6(a,x,q,q)\right]^{\frac1q}\right\}.
\end{aligned}
$$

5 Comments

The main contributions of this paper has been establish some new Ostrowski and Simpson type inequalities for the class of strongly reciprocally convex functions. We expect that the ideas and techniques used in this paper may inspire interested readers in to explore some new applications of these functions in various fields of pure and applied sciences.

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