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Some Families of Analytic Functions in the Upper Half-Plane and Their Associated Differential Subordination and Differential Superordination Properties and Problems

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Abstract: The existing literature in Geometric Function Theory of Complex Analysis contains a considerably large number of interesting investigations dealing with differential subordination and differential superordination problems for analytic functions in the unit disk. Nevertheless, only a few of these earlier investigations deal with the above-mentioned problems in the upper half-plane. The notion of differential subordination in the upper half-plane was introduced by Răducanu and Pascu in [16]. For a set Ω in the complex plane $\mathbb{C}$, let the function $p(z)$ be analytic in the upper half-plane Δ given by

$$\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad \Im(z) > 0\}$$

and suppose that $\psi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. The main object of this article is to consider the problem of determining properties of functions $p(z)$ that satisfy the following differential superordination:

$$\Omega \subset \left\{ \psi\big(p(z), p'(z), p''(z); z\big) : z \in \Delta \right\}.$$

We also present several applications of the results derived in this article to differential subordination and differential superordination for analytic functions in Δ.

Keywords: Analytic functions; Univalent functions; Starlike functions; Convex functions; Upper half-plane; Differential subordination; Differential superordination; Admissible functions.

1 Introduction

Let Δ denote the upper half-plane, that is,

$$\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad \Im(z) > 0\},$$

and let $\mathscr{H}[\Delta]$ denote the class of functions $f : \Delta \to \mathbb{C}$ which are analytic in Δ and which satisfy the so-called hydrodynamic normalization (see [1], [15] and [20])

$$\lim_{\Delta \ni z \to \infty} [f(z) - z] = 0.$$

Also let $\mathscr{S}[\Delta]$ denote the class of all functions in $\mathscr{H}[\Delta]$ which are univalent in Δ. Various basic properties concerning functions belonging to the class $\mathscr{S}[\Delta]$ were developed in a series of articles (see, for details, [11], [21] and [22]).

A function $f \in \mathscr{H}[\Delta]$, with $f(z) \neq 0$, is said to be starlike in Δ if and only if

$$\Im\left(\frac{f'(z)}{f(z)}\right) < 0 \qquad (z \in \Delta).$$

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We denote by $\mathscr{S}^*[\Delta]$ the subclass of $\mathscr{H}[\Delta]$ consisting of functions which are starlike in Δ. We note that, the functions in the class $\mathscr{S}^*[\Delta]$ have the property $0 \notin f(\Delta)$.

A function $f \in \mathscr{H}[\Delta]$, with $f(z) \neq z$ and $f'(z) \neq 0$, is said to be convex in Δ if and only if

$$\Im\left(\frac{f''(z)}{f'(z)}\right) > 0 \qquad (z \in \Delta).$$

We denote by $\mathscr{K}[\Delta]$ the subclass of $\mathscr{H}[\Delta]$ consisting of functions which are convex in Δ. The classes $\mathscr{S}^*[\Delta]$ and $\mathscr{K}[\Delta]$ were introduced by Stankiewicz [20].

We first need to recall the notion of subordination in the upper half-plane.

Let f and g be members of the class $\mathscr{H}[\Delta]$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a function $\varphi \in \mathscr{H}[\Delta]$, with $\varphi[\Delta] \subset \Delta$, such that

$$f(z) = g\big(\varphi(z)\big) \qquad (z \in \Delta).$$

In such a case, we write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \qquad (z \in \Delta).$$

Furthermore, if the function g is univalent in Δ, then we have the following equivalence (*cf.* [16]):

$$f(z) \prec g(z) \quad (z \in \Delta) \quad \Longleftrightarrow \quad f(\Delta) \subset g(\Delta).$$

Let Ω be any set in the complex plane $\mathbb{C}$. Also let p be analytic in Δ and suppose that $\psi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. Răducanu and Pascu [16] extended the theory of differential subordination to the upper half-plane by using methods similar to those used in the unit disk introduced by Miller and Mocanu [13]. They determined properties of functions p that satisfy the following differential subordination:

$$\big\{\psi\big(p(z), p'(z), p''(z); z\big) : z \in \Delta\big\} \subset \Omega.$$

We will now recall some definitions and a theorem, which are required in our present investigation.

Definition 1 (see [13, p. 403, Definition 8.3i]). Denote by $\mathscr{Q}(\Delta)$ the set of functions $q \in \mathscr{H}[\Delta]$ that are analytic and injective on $\overline{\Delta} \setminus E(q)$, where

$$E(q) = \left\{\xi \in \partial\Delta : \lim_{z \to \xi} q(z) = \infty\right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial\Delta \setminus E(q)$.

Definition 2 (see [16]). Let Ω be a set in $\mathbb{C}$ and $q \in \mathscr{Q}(\Delta)$. The class of admissible functions $\Psi_\Delta[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t; z) \notin \Omega$$

whenever

$$r = q(\xi), \quad s = kq'(\xi)$$

and

$$\Im\left(\frac{t}{q'(\xi)}\right) \geqq k^2\, \Im\left(\frac{q''(\xi)}{q'(\xi)}\right),$$

where $z \in \Delta$, $\xi \in \partial\Delta \setminus E(q)$ and $k \geqq 0$.

If $\psi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$, then the above admissibility condition reduces to the following form:

$$\psi\big(q(\xi), kq'(\xi); z\big) \notin \Omega,$$

where $z \in \Delta$, $\xi \in \partial\Delta \setminus E(q)$ and $k \geqq 0$.

Theorem 1 (see [16]). *Let $\psi \in \Psi_\Delta[\Omega, q]$ and $p \in \mathscr{H}[\Delta]$. If*

$$\psi\big(p(z), p'(z), p''(z); z\big) \in \Omega \qquad (z \in \Delta),$$

then

$$p(z) \prec q(z) \qquad (z \in \Delta).$$

In this sequel to the recent paper [24], we follow the theory of differential superordinations in the unit disk, which was introduced by Miller and Mocanu [14], and consider the dual problem of determining properties of functions p that satisfy the following differential superordination:

$$\Omega \subset \big\{\psi\big(p(z), p'(z), p''(z); z\big) : z \in \Delta\big\}.$$

In other words, we determine the conditions on Ω, Σ and ψ for which the following implication holds true:

$$\begin{aligned} &\Omega \subset \big\{\psi\big(p(z), p'(z), p''(z); z\big) : z \in \Delta\big\} \\ &\quad \Longrightarrow \Sigma \subset p(\Delta), \end{aligned} \tag{1.1}$$

where Σ is any set in $\mathbb{C}$. The results presented in this paper would provide improvements and generalizations of these in the aforementioned work [24].

If either Ω or Σ is a simply-connected domain, then (1.1) can be rephrased in terms of superordination. If p is univalent in Δ, and if Σ is a simply-connected domain with $\Sigma \neq \mathbb{C}$, then there is a conformal mapping q of Δ onto Σ such that $q(0) = p(0)$. In this case, (1.1) can be rewritten as follows:

$$\begin{aligned} &\Omega \subset \big\{\psi\big(p(z), p'(z), p''(z); z\big) : z \in \Delta\big\} \\ &\quad \Longrightarrow \quad q(z) \prec p(z) \qquad (z \in \Delta). \end{aligned} \tag{1.2}$$

If Ω is also a simply-connected domain with $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of Δ onto Ω such that $h(0) = \psi\big(p(0), 0, 0; 0\big)$. In addition, if the function

$\psi\big(p(z), p'(z), p''(z); z\big)$ is univalent in Δ, then (1.2) can be rewritten as follows:

$$h(z) \prec \psi\big(p(z), p'(z), p''(z); z\big) \qquad (z \in \Delta)$$
$$\Longrightarrow \; q(z) \prec p(z) \qquad (z \in \Delta).$$

There are three key ingredients in the implication relationship (1.2): the differential operator ψ, the set Ω and the *dominating* function q. If two of these entities were given, one would hope to find conditions on the third entity so that (1.2) would be satisfied. In this article, we start with a given set Ω and a given function q, and we then determine a set of "admissible" operators ψ so that (1.2) holds true.

We first introduce the following definition.

Definition 3. Let $\psi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ and the function h be analytic in Δ. If the functions p and $\psi\big(p(z), p'(z), p''(z); z\big)$ are univalent in Δ and satisfy the following (second-order) differential superordination:

$$h(z) \prec \psi\big(p(z), p'(z), p''(z); z\big) \qquad (z \in \Delta), \qquad (1.3)$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solution of the differential superordination or, more simply, a subordinant if $q \prec p$ for p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies the following condition:

$$q(z) \prec \widetilde{q}(z) \qquad (z \in \Delta)$$

for all subordinants q of (1.3) is said to be the best subordinant. We note that the best subordinant is unique up to a rotation of Δ.

For Ω a set in $\mathbb{C}$, with ψ and p as given in Definition 3, we suppose that (1.3) is replaced by

$$\Omega \subset \big\{\psi\big(p(z), p'(z), p''(z); z\big) : z \in \Delta\big\}.$$

Although this more general situation is a "differential containment", yet we also refer to it as a differential superordination, and the definitions of solution, subordinant and best subordinant as given above can be extended to this more general case (see also the recent works [8] and [12]).

We will use the following lemma [13, p. 405, Lemma 8.3k] from the theory of differential subordinations in Δ to determine subordinants of the differential superordinations in Δ.

Lemma (see [13]). *Let $q \in \mathscr{H}[\Delta]$ and $p \in \mathscr{Q}(\Delta)$. If q is not subordinate to p, then there exist points $z_0 = x_0 + iy_0 \in \Delta$ and $\xi_0 \in \partial\Delta \setminus E(p)$, and an $m > 0$, such that*

(i) $q(z_0) = p(\xi_0)$,

(ii) $q(\Delta_0) \subset p(\Delta)$, *where*

$$\Delta_{y_0} = \{z : z \in \mathbb{C} \quad \text{and} \quad \Im(z) > y_0\},$$

(iii) $q'(z_0) = mp(\xi_0)$

and

(iv) $\Im\left(\dfrac{q''(z_0)}{q'(z_0)}\right) \geqq m^2\, \Im\left\{\dfrac{p''(\xi_0)}{q'(z_0)}\right\}$.

2 A Class of Admissible Functions and a Fundamental Result

In this section, we first define the class of admissible functions referred to in the preceding section.

Definition 4. Let Ω be a set in $\mathbb{C}$ and $q \in \mathscr{H}[\Delta]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_{\Delta}[\Omega, q]$ consists of those functions ψ given by $\psi : \mathbb{C}^3 \times \overline{\Delta} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t; \xi) \in \Omega$$

whenever

$$r = q(z), \qquad s = \frac{q'(z)}{m}$$

and

$$\Im\left(\frac{t}{q'(z)}\right) \leqq \frac{1}{m^2}\, \Im\left(\frac{q''(z)}{q'(z)}\right), \qquad (2.1)$$

where $z \in \Delta$, $\xi \in \partial\Delta$ and $m > 0$.

If $\psi : \mathbb{C}^2 \times \overline{\Delta} \to \mathbb{C}$, then the admissible condition (2.1) reduces to following form:

$$\psi\left(q(z), \frac{q'(z)}{m};\, \xi\right) \in \Omega \qquad (z \in \Delta;\ \xi \in \partial\Delta;\ m > 0).$$

The next theorem is a foundation result in the theory of the first-order and the second-order differential superordinations in Δ.

Theorem 2. *Let $\psi \in \Psi'_{\Delta}[\Omega, q]$ and $q \in \mathscr{H}[\Delta]$. If $p \in \mathscr{Q}(\Delta)$ and $\psi\big(p(z), p'(z), p''(z); z\big)$ is univalent in Δ, then*

$$\Omega \subset \big\{\psi\big(p(z), p'(z), p''(z); z\big) : z \in \Delta\big\} \qquad (2.2)$$

implies that

$$q(z) \prec p(z) \qquad (z \in \Delta).$$

Proof. Suppose that

$$q(z) \not\prec p(z) \qquad (z \in \Delta).$$

Then, by the above Lemma, there exist points $z_0 = x_0 + iy_0 \in \Delta$ and $\xi_0 \in \partial\Delta \setminus E(p)$, and an $m > 0$. that satisfy the conditions (i) to (iv) of the above Lemma. Using these conditions with $r = p(\xi_0)$, $s = p'(\xi_0)$, $t = p''(\xi_0)$ and $\xi = \xi_0$ in Definition 4, we obtain

$$\psi\big(p(\xi_0), p'(\xi_0), p''(\xi_0); \xi_0\big) \in \Omega,$$

which contradicts (2.2), so we have

$$q(z) \prec p(z) \qquad (z \in \Delta).$$

In the special case when $\Omega \neq \mathbb{C}$ is a simply-connected domain and h is a conformal mapping of Δ onto Ω, we denote this class $\Psi'_\Delta[h(\Delta), q]$ by $\Psi'_\Delta[h, q]$. The following result is an immediate consequence of Theorem 2.

Theorem 3. *Let $q \in \mathscr{H}[\Delta]$. Also let the function h be analytic in Δ and suppose that $\psi \in \Psi'_\Delta[h, q]$. If $p \in \mathscr{Q}(\Delta)$ and $\psi\big(p(z), p'(z), p''(z); z\big)$ is univalent in Δ, then*

$$h(z) \prec \psi\big(p(z), p'(z), p''(z); z\big) \qquad (z \in \Delta) \tag{2.3}$$

implies that

$$q(z) \prec p(z) \qquad (z \in \Delta).$$

Theorems 2 and 3 can only be used to obtain the subordinants of the differential superordination of the form (2.2) or (2.3).

Theorem 4. *Let the function h be analytic in Δ and let $\psi : \mathbb{C}^3 \times \overline{\Delta} \to \mathbb{C}$. Suppose that the following differential equation*:

$$\psi\big(q(z), q'(z), q''(z); z\big) = h(z) \tag{2.4}$$

has a solution $q \in \mathscr{Q}(\Delta)$. If

$$\psi \in \Psi'_\Delta[h, q], \quad p \in \mathscr{Q}(\Delta)$$

and

$$\psi\big(p(z), p'(z), p''(z); z\big)$$

is univalent in Δ, then (2.3) *implies that*

$$q(z) \prec p(z) \qquad (z \in \Delta)$$

and q is the best subordinant.

Proof. Since $\psi \in \Psi'_\Delta[h, q]$, by applying Theorem 3, we deduce that q is a subordinant of (2.3). Since q satisfies (2.4), it is also a solution of the differential superordination (2.3). Therefore, all subordinants of (2.3) will be subordinate to q. It follows that q will be the best subordinant of (2.3).

In the next two sections, by making use of the differential subordination results of Răducanu and Pascu [16] in the upper half-plane Δ and the differential superordination results in Δ obtained in Section 2 (see, for details, Theorems 2, 3 and 4), we determine certain appropriate classes of admissible functions and investigate some differential subordination and differential superordination properties of analytic functions in Δ. It should be remarked in passing that, in recent years, several authors obtained many interesting results associated with differential subordination and differential superordination in the unit disk. The interested reader may refer to several earlier works including (for example) [2] to [10], [17], [18], [19], [23], [25] to [27], [29] and [30] (see also [28] for some applications of differential subordination and *strong* differential subordination in Probability Theory).

3 A Useful Set of Subordination Results

We first define the following class of admissible functions that are required in proving our first result.

Definition 5. Let Ω be a set in $\mathbb{C}$ and let $q \in \mathscr{Q}(\Delta)$. The class $\Phi_\Delta[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\xi), \quad v = \frac{kq'(\xi)}{q(\xi)} \qquad \big(q(\xi) \neq 0\big),$$

and

$$\Im\left(\frac{u(wv + v^2)}{q'(\xi)}\right) \geqq k^2\, \Im\left(\frac{q''(\xi)}{q'(\xi)}\right)$$
$$\big(z \in \Delta;\ \xi \in \partial\Delta \setminus E(q);\ k \geqq 0\big).$$

Theorem 5. *Let $\phi \in \Phi_\Delta[\Omega, q]$, $f(z) \neq 0$ and $f'(z) \neq 0$. If $f \in \mathscr{H}[\Delta]$ satisfies the following condition*:

$$\left\{\phi\left(\frac{f(z)}{f'(z)}, \frac{f'(z)}{f(z)} - \frac{f''(z)}{f'(z)}, \frac{f(z)\big[[f''(z)]^2 - f'(z)f'''(z)\big]}{f'(z)\big[[f'(z)]^2 - f(z)f''(z)\big]} - \frac{f'(z)}{f(z)}; z\right) : z \in \Delta\right\} \subset \Omega, \tag{3.1}$$

then

$$\frac{f(z)}{f'(z)} \prec q(z) \qquad (z \in \Delta).$$

Proof. Define the function $p(z)$ in Δ by

$$p(z) = \frac{f(z)}{f'(z)} \qquad (z \in \Delta). \tag{3.2}$$

A simply calculation yields

$$\frac{f'(z)}{f(z)} - \frac{f''(z)}{f'(z)} = \frac{p'(z)}{p(z)} \qquad (z \in \Delta). \tag{3.3}$$

Further computations show that

$$\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)}=\frac{p''(z)}{p'(z)}-\frac{p'(z)}{p(z)}. \tag{3.4}$$

We now define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r,s,t)=r, \qquad v(r,s,t)=\frac{s}{r}$$

and

$$w(r,s,t)=\frac{rt-s^2}{rs}. \tag{3.5}$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z)=\phi\left(r,\frac{s}{r},\frac{rt-s^2}{rs};z\right). \tag{3.6}$$

Using equations (3.2) to (3.4), we find from (3.6) that

$$\psi\big(p(z),p'(z),p''(z);z\big)$$

$$=\phi\left(\frac{f(z)}{f'(z)},\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)},\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)};z\right). \tag{3.7}$$

Therefore, (3.1) becomes

$$\psi\big(p(z),p'(z),p''(z);z\big)\in\Omega.$$

We easily find from (3.5) that

$$t=u(wv+v^2). \tag{3.8}$$

Thus, clearly, the admissibility condition for $\phi\in\Phi_\Delta[\Omega,q]$ in Definition 5 is equivalent to the admissibility condition for ψ as given in Definition 2. Therefore, we have $\psi\in\Psi_\Delta[\Omega,q]$ and, by Theorem 1, we get

$$p(z)\prec q(z) \qquad (z\in\Delta)$$

or, equivalently,

$$\frac{f(z)}{f'(z)}\prec q(z) \qquad (z\in\Delta),$$

which evidently completes the proof of Theorem 5.

If $\Omega\neq\mathbb{C}$ is a simply-connected domain, then $\Omega=h(\Delta)$ for some conformal mapping $h(z)$ of Δ onto Ω. In this case, the class $\Phi_\Delta[h(\Delta),q]$ is written (for convenience) as $\Phi_\Delta[h,q]$. The following result is an immediate consequence of Theorem 5.

Theorem 6. *Let $\phi\in\Phi_\Delta[h,q]$, $f(z)\neq 0$ and $f'(z)\neq 0$. If $f\in\mathscr{H}[\Delta]$ satisfies the following condition:*

$$\phi\left(\frac{f(z)}{f'(z)},\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)},\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)};z\right)\prec h(z)$$

$$(z\in\Delta), \tag{3.9}$$

then

$$\frac{f(z)}{f'(z)}\prec q(z) \qquad (z\in\Delta).$$

Our next result is an extension of Theorem 5 to the case when the behavior of $q(z)$ on $\partial\Delta$ is not known.

Theorem 7. *Let the functions h and q be univalent in Δ with $q\in\mathscr{Q}(\Delta)$ and set $q_\rho(z)=q(\rho z)$ and $h_\rho(z)=h(\rho z)$. Suppose also that $\phi:\mathbb{C}^3\times\Delta\to\mathbb{C}$ satisfies one of the following conditions:*

(1) *$\phi\in\Phi_\Delta[h,q_\rho]$ for some $\rho\in(0,1)$*

or

(2) *There exists $\rho_0\in(0,1)$ such that $\phi\in\Phi_\Delta[h_\rho,q_\rho]$ for all $\rho\in(\rho_0,1)$.*

If $f\in\mathscr{H}[\Delta]$ satisfies (3.9), then

$$\frac{f(z)}{f'(z)}\prec q(z) \qquad (z\in\Delta).$$

Proof. The proof of Theorem 7 is similar to that of a known result [13, p. 30, Theorem 2.3d] and so we choose to omit it.

Our next theorem yields the best dominant of the differential subordination (3.9).

Theorem 8. *Let the function h be univalent in Δ and let $\phi:\mathbb{C}^3\times\Delta\to\mathbb{C}$. Suppose that the following differential equation:*

$$\phi\left(q(z),\frac{q'(z)}{q(z)},\frac{q''(z)}{q'(z)}-\frac{q'(z)}{q(z)};z\right)=h(z) \tag{3.10}$$

has a solution $q(z)$ and satisfies one of the following conditions:

(1) *$q\in\mathscr{Q}(\Delta)$ and $\phi\in\Phi_\Delta[h,q]$,*

(2) *q is univalent in Δ and $\phi\in\Phi_\Delta[h,q_\rho]$ for some $\rho\in(0,1)$*

or

(3) *q is univalent in Δ and there exists $\rho_0\in(0,1)$ such that $\phi\in\Phi_\Delta[h_\rho,q_\rho]$ for all $\rho\in(\rho_0,1)$.*

If $f\in\mathscr{H}[\Delta]$ satisfies (3.9), then

$$\frac{f(z)}{f'(z)}\prec q(z) \qquad (z\in\Delta)$$

and q is the best dominant.

Proof. Following the same arguments as for proving the known result [13, p. 31, Theorem 2.3e], we deduce that q is a dominant from Theorems 6 and 7. Since q satisfies (3.10), it is also a solution of (3.9) and, therefore, q will be dominated by all dominants. Hence q is the best dominant.

In view of Definition 5, in the particular case when $q(z) = z$, the class $\Phi_\Delta[\Omega, q]$ of admissible functions, denoted simply by $\Phi_\Delta[\Omega, z]$, is described below.

Definition 6. Let Ω be a set in $\mathbb{C}$. The class $\Phi_\Delta[\Omega, z]$ of admissible functions consists of those functions $\phi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ such that

$$\phi\left(\eta, \frac{k}{\eta}, \frac{L\eta - k^2}{k\eta}; z\right) \notin \Omega \tag{3.11}$$

whenever $z \in \Delta$, $\Im(L) \geqq 0$, $\eta \in \mathbb{R} \setminus \{0\}$ and $k > 0$

Corollary 1. *Let $\phi \in \Phi_\Delta[\Omega, z]$, $f(z) \neq 0$ and $f'(z) \neq 0$. If $f \in \mathscr{H}[\Delta]$ satisfies the following condition*:

$$\phi\left(\frac{f(z)}{f'(z)}, \frac{f'(z)}{f(z)} - \frac{f''(z)}{f'(z)}, \frac{f(z)\left[[f''(z)]^2 - f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2 - f(z)f''(z)\right]} - \frac{f'(z)}{f(z)}; z\right) \in \Omega,$$

then

$$\frac{f(z)}{f'(z)} \prec z \qquad (z \in \Delta).$$

In the special case when

$$\Omega = q(\Delta) = \{\omega : \Im(\omega) > 0\},$$

the class $\Phi_\Delta[\Omega, z]$ is denoted, for brevity, by $\Phi_\Delta[\Delta, z]$. Corollary 1 can now be rewritten in the following form.

Corollary 2. *Let $\phi \in \Phi_\Delta[\Delta, z]$, $f(z) \neq 0$ and $f'(z) \neq 0$. If $f \in \mathscr{H}[\Delta]$ satisfies the following condition*:

$$\Im\left[\phi\left(\frac{f(z)}{f'(z)}, \frac{f'(z)}{f(z)} - \frac{f''(z)}{f'(z)}, \frac{f(z)\left[[f''(z)]^2 - f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2 - f(z)f''(z)\right]} - \frac{f'(z)}{f(z)}; z\right)\right] > 0,$$

then

$$\Im\left(\frac{f(z)}{f'(z)}\right) > 0 \qquad (z \in \Delta).$$

Example 1. Let the functions $A : \Delta \to \mathbb{C}$ and $B : \Delta \to \mathbb{C}$ be analytic in Δ and satisfy $\Im[A(z)] \leqq 0$ and $\Im[B(z)] \leqq 0$. Then the functions

$$\phi_1(u, v, w; z) = \frac{1}{u} - v + A(z) \qquad \text{and} \qquad \phi_2(u, v, w; z) = vw + B(z)$$

satisfy the admissibility condition (3.11). Hence Corollary 1 yields

$$\Im\left(\frac{f''(z)}{f'(z)} + A(z)\right) > 0 \implies \Im\left(\frac{f(z)}{f'(z)}\right) > 0$$

and

$$\Im\left(\left(\frac{f'(z)}{f(z)} - \frac{f''(z)}{f'(z)}\right)' + B(z)\right) > 0 \implies \Im\left(\frac{f(z)}{f'(z)}\right) > 0.$$

We next introduce the following class of admissible functions.

Definition 7. Let Ω be a set in $\mathbb{C}$ and $q \in \mathscr{Q}(\Delta)$. The class $\Phi_{\Delta,1}[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi\left(q(\xi), kq'(\xi); z\right) \notin \Omega, \tag{3.12}$$

where $z \in \Delta$, $\xi \in \partial\Delta \setminus E(q)$ and $k \geqq 0$.

Theorem 9. *Let $\phi \in \Phi_{\Delta,1}[\Omega, q]$ and $f'(z) \neq 0$. If $f \in \mathscr{H}[\Delta]$ satisfies the following condition*:

$$\left\{\phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z) \cdot f''(z)}{[f'(z)]^2}; z\right) : z \in \Delta\right\} \subset \Omega, \tag{3.13}$$

then

$$\frac{f(z)}{f'(z)} \prec q(z) \qquad (z \in \Delta).$$

Proof. Define the function $p(z)$ in Δ by

$$p(z) = \frac{f(z)}{f'(z)} \qquad (z \in \Delta). \tag{3.14}$$

A simply calculation yields

$$1 - \frac{f(z) \cdot f''(z)}{[f'(z)]^2} = p'(z). \tag{3.15}$$

We next define the transformation from $\mathbb{C}^2$ to $\mathbb{C}$ by

$$u(r, s) = r \qquad \text{and} \qquad v(r, s) = s. \tag{3.16}$$

Then, upon setting

$$\psi(r, s; z) = \phi(u, v; z) = \phi\,(r, s; z)\,, \tag{3.17}$$

the proof will make use of Theorem 1. Indeed, if we use the equations (3.14) and (3.15), we find from (3.17) that

$$\begin{aligned} &\psi\left(p(z), p'(z); z\right) \\ &\quad = \phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z) \cdot f''(z)}{[f'(z)]^2}; z\right), \end{aligned} \tag{3.18}$$

so that (3.13) becomes

$$\psi\left(p(z), p'(z); z\right) \in \Omega.$$

We now see from (3.17) that the admissibility condition for $\phi \in \Phi_{\Delta,1}[\Omega, q]$ in Definition 7 is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi_\Delta[\Omega, q]$ and, by Theorem 1, we have

$$p(z) \prec q(z) \qquad (z \in \Delta)$$

or, equivalently,

$$\frac{f(z)}{f'(z)} \prec q(z) \qquad (z \in \Delta).$$

We will denote the class $\Phi_{\Delta,1}[h(\Delta),q]$ by $\Phi_{\Delta,1}[h,q]$, where h is the conformal mapping of Δ onto $\Omega \neq \mathbb{C}$.

Theorem 10. *Let $\phi \in \Phi_{\Delta,1}[h,q]$ and $f'(z) \neq 0$. If $f \in \mathscr{H}[\Delta]$ satisfies the following condition*:

$$\phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z)\cdot f''(z)}{[f'(z)]^2}; z\right) \prec h(z) \qquad (z \in \Delta), \tag{3.19}$$

then

$$\frac{f(z)}{f'(z)} \prec q(z) \qquad (z \in \Delta). \tag{3.20}$$

We extend Theorem 10 to the case in which the behavior of $q(z)$ on $\partial\Delta$ is not known.

Theorem 11. *Let $\Omega \subset \mathbb{C}$ and let the function q be univalent in Δ with $q \in \mathscr{Q}(\Delta)$. Suppose also that $\phi \in \Phi_{\Delta,1}[h,q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathscr{H}[\Delta]$ satisfies* (3.13), *then* (3.20) *holds true.*

As a special case, when $q(z) = z$, we get the following corollary.

Corollary 3. *Let Ω be a set in $\mathbb{C}$ and $\phi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ satisfy the following condition*:

$$\phi(\eta, k; z) \notin \Omega \tag{3.21}$$

whenever $z \in \Delta$, $\eta \in \mathbb{R}$ and $k \geqq 0$. If $f \in \mathscr{H}[\Delta]$, with $f'(z) \neq 0$, satisfies the following condition:

$$\phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z)\cdot f''(z)}{[f'(z)]^2}; z\right) \in \Omega,$$

then

$$\Im\left(\frac{f(z)}{f'(z)}\right) > 0 \qquad (z \in \Delta).$$

In the special case when

$$\Omega = q(\Delta) = \{\omega : \Im(\omega) > 0\},$$

Corollary 3 can thus be restated as follows.

Corollary 4. *Let $\phi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ satisfy the following inequality*:

$$\Im[\phi(\eta, k; z)] \leqq 0 \qquad (z \in \Delta)$$

whenever $z \in \Delta$, $\eta \in \mathbb{R}$ and $k \geqq 0$. If $f \in \mathscr{H}[\Delta]$, with $f'(z) \neq 0$, satisfies the following condition:

$$\Im\left(\phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z)\cdot f''(z)}{[f'(z)]^2}; z\right)\right) > 0 \qquad (z \in \Delta),$$

then

$$\Im\left(\frac{f(z)}{f'(z)}\right) > 0 \qquad (z \in \Delta).$$

Example 2. Let the function $D : \Delta \to \mathbb{C}$ be analytic in Δ and satisfy the following inequality:

$$\Im[D(z)] \leqq 0 \qquad (z \in \Delta).$$

Then the function

$$\phi(u, v; z) = u + v + D(z)$$

satisfies the admissibility condition (3.21). Hence Corollary 4 becomes

$$\Im\left(1 + \frac{f(z)}{f'(z)} - \frac{f(z)\cdot f''(z)}{[f'(z)]^2} + D(z)\right) > 0$$
$$\Longrightarrow \Im\left(\frac{f(z)}{f'(z)}\right) > 0 \qquad (z \in \Delta).$$

4 Differential Superordinations and Sandwich-Type Results

In this section, we investigate the dual problem of differential subordination (that is, differential superordination) in the upper half-plane. Because of this, the class of admissible functions is given in the following definition.

Definition 8. Let Ω be a set in $\mathbb{C}$ and $q \in \mathscr{H}[\Delta]$ with $q'(z) \neq 0$. The class $\Phi'_\Delta[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^3 \times \overline{\Delta} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, w; \xi) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{q'(z)}{mq(z)} \qquad (q(z) \neq 0),$$

and

$$\Im\left(\frac{u(wv + v^2)}{q'(z)}\right) \leqq \frac{1}{m^2}\,\Im\left(\frac{q''(z)}{q'(z)}\right)$$
$$(z \in \Delta;\ \xi \in \partial\Delta;\ m > 0).$$

Theorem 12. *Let $\phi \in \Phi'_\Delta[\Omega, q]$, $f(z) \neq 0$ and $f'(z) \neq 0$. If $f \in \mathscr{H}[\Delta]$,*

$$\frac{f(z)}{f'(z)} \in \mathscr{Q}(\Delta)$$

and

$$\phi\left(\frac{f(z)}{f'(z)},\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)},\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)};z\right)$$

is univalent in Δ, *then*

$$\Omega\subset\left\{\phi\left(\frac{f(z)}{f'(z)},\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)},\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)};z\right):z\in\Delta\right\}, \tag{4.1}$$

then

$$q(z)\prec\frac{f(z)}{f'(z)}\qquad(z\in\Delta).$$

Proof. Let the function $p(z)$ be defined by (3.2) and ψ by (3.6). Since $\phi\in\Phi'_\Delta[\Omega,q]$, (3.7) and (4.1) yield

$$\Omega\subset\left\{\psi\big(p(z),p'(z),p''(z);z\big):z\in\Delta\right\}.$$

We see from (3.5) that the admissible condition for $\phi\in\Phi'_\Delta[\Omega,q]$ in Definition 8 is equivalent to the admissible condition for ψ as given in Definition 4. Hence $\psi\in\Psi'_\Delta[\Omega,q]$ and, by Theorem 2, we have

$$q(z)\prec p(z)\qquad(z\in\Delta)$$

or, equivalently,

$$q(z)\prec\frac{f(z)}{f'(z)}\qquad(z\in\Delta),$$

which evidently completes the proof of Theorem 12.

If $\Omega\neq\mathbb{C}$ is a simply-connected domain and $\Omega=h(\Delta)$ for some conformal mapping $h(z)$ of Δ onto Ω, then the class $\Phi'_\Delta[h(\Delta),q]$ is written simply as $\Phi'_\Delta[h,q]$. Proceeding similarly as in the preceding section, the following result is an immediate consequence of Theorem 12.

Theorem 13. *Let* $q\in\mathscr{H}[\Delta]$. *Also let the function* h *be analytic in* Δ *and* $\phi\in\Phi'_\Delta[h,q]$. *If* $f\in\mathscr{H}[\Delta]$, *with* $f(z)\neq0$ *and* $f'(z)\neq0$,

$$\frac{f(z)}{f'(z)}\in\mathscr{Q}(\Delta)$$

and

$$\phi\left(\frac{f(z)}{f'(z)},\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)},\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)};z\right)$$

is univalent in Δ, *then*

$$h(z)\prec\phi\left(\frac{f(z)}{f'(z)},\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)},\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)};z\right)$$

$$(z\in\Delta) \tag{4.2}$$

implies that

$$q(z)\prec\frac{f(z)}{f'(z)}\qquad(z\in\Delta).$$

Theorems 12 and 13 can only be used to obtain subordinations involving the differential superordination of the form (4.1) or (4.2). The following theorem proves the existence of the best subordinant of (4.2) for a suitably chosen ϕ.

Theorem 14. *Let the function* h *be analytic in* Δ *and let* $\phi:\mathbb{C}^3\times\overline{\Delta}\to\mathbb{C}$. *Suppose that the following differential equation:*

$$\phi\left(q(z),\frac{q'(z)}{q(z)},\frac{q''(z)}{q'(z)}-\frac{q'(z)}{q(z)};z\right)=h(z)$$

has a solution $q\in\mathscr{Q}(\Delta)$. *If* $\phi\in\Phi'_\Delta[h,q]$, $f\in\mathscr{H}[\Delta]$, *with* $f(z)\neq0$ *and* $f'(z)\neq0$,

$$\frac{f(z)}{f'(z)}\in\mathscr{Q}(\Delta)$$

and

$$\phi\left(\frac{f(z)}{f'(z)},\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)},\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)};z\right)$$

is univalent in Δ, *then*

$$h(z)\prec\phi\left(\frac{f(z)}{f'(z)},\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)},\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)};z\right)$$

$$(z\in\Delta)$$

implies that

$$q(z)\prec\frac{f(z)}{f'(z)}\qquad(z\in\Delta),$$

and $q(z)$ *is the best subordinant.*

Proof. The proof of Theorem 14 is similar to that of Theorem 8 and it is being omitted here.

By combining Theorems 6 and 13, we obtain the following sandwich-type result.

Corollary 5. *Let the functions* h_1 *and* q_1 *be analytic in* Δ. *Also let the function* h_2 *be in* Δ, $q_2\in\mathscr{Q}(\Delta)$ *and*

$$\phi\in\Phi_\Delta[h_2,q_2]\cap\Phi'_\Delta[h_1,q_1].$$

If $f\in\mathscr{H}[\Delta]$, *with* $f(z)\neq0$ *and* $f'(z)\neq0$,

$$\frac{f(z)}{f'(z)}\in\mathscr{Q}(\Delta)$$

and

$$\phi\left(\frac{f(z)}{f'(z)},\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)},\frac{f(z)\left[[f''(z)]^2-f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2-f(z)f''(z)\right]}-\frac{f'(z)}{f(z)};z\right)$$

is univalent in Δ, *then*

$$h_1(z) \prec \phi\left(\frac{f(z)}{f'(z)}, \frac{f'(z)}{f(z)} - \frac{f''(z)}{f'(z)}, \frac{f(z)\left[[f''(z)]^2 - f'(z)f'''(z)\right]}{f'(z)\left[[f'(z)]^2 - f(z)f''(z)\right]} - \frac{f'(z)}{f(z)}; z\right)$$
$$\prec h_2(z) \qquad (z \in \Delta)$$

implies that

$$q_1(z) \prec \frac{f(z)}{f'(z)} \prec q_2(z) \qquad (z \in \Delta).$$

Definition 9. Let Ω be a set in $\mathbb{C}$ and $q \in \mathscr{H}[\Delta]$. The class $\Phi'_{\Delta,1}[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^2 \times \overline{\Delta} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi\left(q(z), \frac{q'(z)}{m}; \xi\right) \in \Omega \qquad (z \in \Delta;\ \xi \in \partial\Delta;\ m > 0).$$

Theorem 15. *Let* $\phi \in \Phi'_{\Delta,1}[\Omega, q]$. *If* $f \in \mathscr{H}[\Delta]$, *with* $f'(z) \neq 0$,

$$\frac{f(z)}{f'(z)} \in \mathscr{Q}(\Delta) \qquad \textit{and} \qquad \phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z) \cdot f''(z)}{[f'(z)]^2}; z\right)$$

is univalent in Δ, *then*

$$\Omega \subset \left\{\phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z) \cdot f''(z)}{[f'(z)]^2}; z\right) : z \in \Delta\right\} \tag{4.3}$$

implies that

$$q(z) \prec \frac{f(z)}{f'(z)} \qquad (z \in \Delta).$$

Proof. Let $p(z)$ be defined by (3.14) and ψ by (3.17). Since $\phi \in \Phi'_{\Delta,1}[\Omega, q]$, it follows from (3.18) and (4.3) that

$$\Omega \subset \left\{\psi\big(p(z), p'(z); z\big) : z \in \Delta\right\}.$$

We know from (3.16) that the admissible condition for $\phi \in \Phi'_{\Delta,1}[\Omega, q]$ in Definition 9 is equivalent to the admissible condition for ψ as given in Definition 4. Hence $\psi \in \Psi'_{\Delta}[\Omega, q]$ and, by Theorem 2, we get

$$q(z) \prec p(z) \qquad (z \in \Delta)$$

or, equivalently,

$$q(z) \prec \frac{f(z)}{f'(z)} \qquad (z \in \Delta).$$

In the case when $\Omega \neq \mathbb{C}$ is a simply-connected domain with $\Omega = h(\Delta)$ for some conformal mapping $h(z)$ of Δ onto Ω, the class $\Phi'_{\Delta,1}[h(\Delta), q]$ is written as $\Phi'_{\Delta,1}[h, q]$. Proceedings similarly, the following result is an immediate consequence of Theorem 15.

Theorem 16. *Let* $q \in \mathscr{H}[\Delta]$ *and the function* h *be analytic in* Δ. *Also let* $\phi \in \Phi'_{\Delta,1}[h, q]$. *If* $f \in \mathscr{H}[\Delta]$, *with* $f'(z) \neq 0$,

$$\frac{f(z)}{f'(z)} \in \mathscr{Q}(\Delta) \qquad \textit{and} \qquad \phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z) \cdot f''(z)}{[f'(z)]^2}; z\right)$$

is univalent in Δ, *then*

$$h(z) \prec \phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z) \cdot f''(z)}{[f'(z)]^2}; z\right) \qquad (z \in \Delta)$$

implies that

$$q(z) \prec \frac{f(z)}{f'(z)} \qquad (z \in \Delta).$$

If we combine Theorems 10 and 16, then we have the following sandwich-type result.

Corollary 6. *Let the functions* h_1 *and* q_1 *be analytic in* Δ. *Also let the function* h_2 *be univalent in* Δ *and suppose that* $q_2 \in \mathscr{Q}(\Delta)$ *and* $\phi \in \Phi_{\Delta,1}[h_2, q_2] \cap \Phi'_{\Delta,1}[h_1, q_1]$. *If* $f \in \mathscr{H}[\Delta]$, *with* $f'(z) \neq 0$,

$$\frac{f(z)}{f'(z)} \in \mathscr{Q}(\Delta) \qquad \textit{and} \qquad \phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z) \cdot f''(z)}{[f'(z)]^2}; z\right)$$

is univalent in Δ, *then*

$$h_1(z) \prec \phi\left(\frac{f(z)}{f'(z)}, 1 - \frac{f(z) \cdot f''(z)}{[f'(z)]^2}; z\right) \prec h_2(z) \qquad (z \in \Delta)$$

implies that

$$q_1(z) \prec \frac{f(z)}{f'(z)} \prec q_2(z) \qquad (z \in \Delta).$$

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