Sohag Journal of Science
An International Journal

http://dx.doi.org/10.18576/sjs/020101

Darboux Ruled Surfaces with Pointwise $1-$Type Gauss Map

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Received: 7 Jun. 2016, Revised: 21 Sep. 2016, Accepted: 23 Sep. 2016
Published online: 1 Jan. 2017

Abstract: The motivation of the present work is to develop the finiteness property in our work [1] by using Frenet formula [2]. Some special cases of ruled surfaces with pointwise $1-$type Gauss map are studied.

Keywords: Ruled surfaces, Gauss map, finite type.

1 Introduction

The study of submanifolds of finite type began in the late 1970's through some author's attempts to find the best possible estimate of the total mean curvature of a compact submanifold of a Euclidean space and to find a notion of "degree" for submanifolds of a Euclidean space. The first results on this subject have been collected in [3] and [4]. Since that time, the subject has had a rapid development.

Although the class of submanifolds of finite type is large, it consists of nice submanifolds of Euclidean spaces. For example, all minimal submanifolds of a Euclidean space and all minimal submanifolds of hyperspheres are of $1-$type and vice versa. Also, all parallel submanifolds of a Euclidean space and all compact homogeneous Riemannian manifolds equivariantly immersed in a Euclidean space are of finite type. Furthermore, similar to minimal submanifolds, finite type submanifolds and finite type maps are characterized by a variational minimal principle in a natural way.

On one hand, the study of finite type submanifolds provides a natural way to combine spectral theory with the geometry of submanifolds and also with the geometry of smooth maps; in particular, with the Gauss map. On the other hand, the tools of geometry of submanifolds can be applied to the study of spectral geometry via the study of finite type submanifolds. The notion of finite type immersion is naturally extended in particular to Gauss map $\mathbf{G}$ on M in Euclidean space [5], such that finite type Gauss map is an especially useful tool in the study of submanifolds [6] and [7]. As is well known, the theory of Gauss map is always one of interesting topics in a Euclidean space and a pseudo-Euclidean space and it has been investigated from the various viewpoints by many differential geometers [8]-[**?**]. Also, Gauss map is important for a variety of applications in computer science as computer image, computer graphics, ... etc.

In this paper, we deal with special cases of ruled surfaces to find the conditions which determine these surfaces of pointwise $1-$type Gauss map of the first kind. Some examples are given.

2 Preliminaries

Let a surface $M:\mathbf{X}=\mathbf{X}(s,v)$ in an Euclidean $3-$space E^3. The map $\mathbf{G}:M\rightarrow S^2(1)\subset E^3$ which sends each point of M to the unit normal vector to M at the point is called the Gauss map of a surface M; where $S^2(1)$ denotes the unit sphere of E^3. Then the Gauss map is given by

$$\mathbf{G}=\frac{\mathbf{X}_s\times\mathbf{X}_v}{\mid\mathbf{X}_s\times\mathbf{X}_v\mid}, \tag{1}$$

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where $\mathbf{X}_s$ and $\mathbf{X}_v$ are the first partial derivatives with respect to the parameters of $\mathbf{X}$. For the matrix (g_{ij}) of the Riemannian metric on M we denote by (g^{ij}) the inverse matrix and g is the determinant of the matrix (g_{ij}). The Laplacian Δ associated with the induced metric g on M is given by

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial X_i} (\sqrt{g}\, g^{ij} \frac{\partial}{\partial X_j}). \tag{2}$$

The mean curvature H of the surface is defined by

$$H = \frac{1}{2} \sum_{i,j=1}^{2} g^{ij} L_{ij}, \tag{3}$$

where L_{ij} are the coefficients of the second fundamental form.

An isometric immersion $\mathbf{X} : M \to E^3$ of a submanifold M in E^3 is said to be of finite type if $\mathbf{X}$ (identified with the position vector field of M in E^3) can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is,

$$\mathbf{X} = \mathbf{X}_0 + \sum_{i=1}^{j} \mathbf{X}_i, \tag{4}$$

where $\mathbf{X}_0$ is a constant map and $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_j$ non-constant maps such that

$$\Delta \mathbf{X}_i = \lambda_i \mathbf{X}_i, \quad \lambda_i \in R, \quad 1 \leq i \leq j \tag{5}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_j$ (eigenvalues) are different, then M is said to be of $j-$type. If in particular, one of λ_i is zero then M is said to be of null $j-$type. Similarly, a smooth map ϕ of a $2-$dimensional Riemannian manifold (surface) M of E^3 is said to be of finite type if ϕ is a finite sum of E^3-valued eigenfunctions of Δ [3] and [4].

Let M be a connected (not necessary compact) surface in E^3. Then the position vector $\mathbf{X}$ and the mean curvature vector H of M in E^3 satisfy

$$\Delta \mathbf{X} = -2\mathbf{H}, \tag{6}$$

where $\mathbf{H} = H\mathbf{G}$ [4]. This formula yields the following well-known result: A surface M in E^3 is minimal if and only if all coordinate functions of E^3, restricted to M, are harmonic functions, that is,

$$\Delta \mathbf{X} = 0. \tag{7}$$

We recall theorem of T.Takahashi [17] and [18] which states that a submanifold M of a Euclidean space is of $1-$type, i.e. the position vector field of the submanifold in the Euclidean space satisfies the differential equation

$$\Delta \mathbf{X} = \lambda \mathbf{X}, \tag{8}$$

for some real number λ, if and only if either the submanifold is a minimal submanifold of the Euclidean space $(\lambda = 0)$ or it is a minimal submanifold of a hypersphere of the Euclidean space centered at the origin $(\lambda \neq 0)$.

If a submanifold M of a Euclidean space has $1-$type Gauss map $\mathbf{G}$, then

$$\Delta \mathbf{G} = \lambda (\mathbf{G} + \mathbf{C}), \quad \text{for some} \quad \lambda \in R, \tag{9}$$

and some constant vector $\mathbf{C}$. However, the Laplacian of Gauss maps of several surfaces such as helicoid, catenoid and right cones in E^3, and also some hypersurfaces has the form of the product

$$\Delta \mathbf{G} = f(\mathbf{G} + \mathbf{C}), \tag{10}$$

for some non-zero smooth real function f on M and some constant vector $\mathbf{C}$. A surface M of a Euclidean space E^3 is said to have pointwise 1-type Gauss map if its Gauss map $\mathbf{G}$ satisfies (10). A pointwise 1-type Gauss map is called proper if the function f defined by (10) is non-constant. A surface with pointwise 1-type Gauss map is said to be first kind if the vector $\mathbf{C}$ in (10) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be second kind cite [7], [19] - [23].

Let $\alpha=\alpha(s): I\subset R\rightarrow E^3$ be an arbitrary curve of arc-length parameter s and I be containing zero. And let $\{\mathbf{t}(s),\mathbf{n}(s),\mathbf{b}(s)\}$ be the moving Frenet frame along α then the Frenet formula is given by

$$\begin{pmatrix}\mathbf{t}'\\ \mathbf{n}'\\ \mathbf{b}'\end{pmatrix}=\begin{pmatrix}0 & k & 0\\ -k & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}, \quad {}'=\frac{d}{ds}, \tag{11}$$

where $\mathbf{t},\mathbf{n}$ and $\mathbf{b}$ are tangent, principal normal and binormal vector along α, respectively, also, k and τ are the curvature and torsion of the curve α, respectively [**?**].

3 Darboux developable ruled surfaces with pointwise $1-$type Gauss map of first kind

Let M_1 be Darboux developable ruled surface which has the parametrization as the following:

$$M_1: \mathbf{X}(s,v)=\mathbf{b}(s)+v\,\mathbf{t}(s), \tag{12}$$

where $\mathbf{t}$ and $\mathbf{b}$ are the tangent and binormal of space curve α respectively. Thus,

$$\mathbf{G}=\mathbf{b}. \tag{13}$$

Hence

$$(g_{ij})=diag\big((\tau-vk)^2,1\big), \quad g=(\tau-vk)^2, \tag{14}$$

where diag(,) is an $2\times 2-$diagonal matrix. Therefore, the formula of the Laplacian Δ takes the following form:

$$\Delta=\frac{vk'-\tau'}{(vk-\tau)^3}\frac{\partial}{\partial s}-\frac{1}{(vk-\tau)^2}\frac{\partial^2}{\partial s^2}-\frac{k}{vk-\tau}\frac{\partial}{\partial v}-\frac{\partial^2}{\partial v^2}; \quad vk\neq\tau. \tag{15}$$

Consequently

$$\Delta\mathbf{G}=\frac{\tau^2}{(vk-\tau)^2}\mathbf{G}-\frac{k\tau}{(vk-\tau)^2}\mathbf{t}+\frac{v(k\tau'-\tau k')}{(vk-\tau)^3}\mathbf{n}. \tag{16}$$

For $\mathbf{G}$ has pointwise $1-$type Gauss map of the first kind, it must investigate the following condition:

$$-\frac{k\tau}{(vk-\tau)^2}\mathbf{t}+\frac{v(k\tau'-\tau k')}{(vk-\tau)^3}\mathbf{n}=0. \tag{17}$$

Since $\mathbf{t}$ and $\mathbf{n}$ are linearly independent. Then,

$$-\frac{k\tau}{(vk-\tau)^2}=0, \quad \frac{v(k\tau'-\tau k')}{(vk-\tau)^3}=0. \tag{18}$$

Solving these simultaneous equations, we get $k=0$ or $\tau=0$. Since,

$$H=\frac{\tau}{2(vk-\tau)}, \quad \Delta\mathbf{x}=\frac{-\tau\mathbf{b}}{vk-\tau}. \tag{19}$$

Then, we get

$$\Delta\mathbf{G}=0, \quad H=0, \quad \Delta\mathbf{x}=0. \tag{20}$$

That is, M_1 is minimal surface of $1-$type and has pointwise $1-$type Gauss map of the first kind.

Theorem 1.*The Darboux developable ruled surface M_1 has pointwise $1-$type Gauss map of the first kind if and only if the base curve of this surface is a plane curve or a straight line.*

*Example 1.*Let α be circle curve and the parametrization of it is

$\alpha=(\cos s,\sin s,0)$.

Then,

$\mathbf{t}=(-\sin s,\cos s,0), \quad \mathbf{b}=(0,0,1)$.

Consequently, the Darboux developable ruled surface is parameterized by

$\mathbf{X}(s,v) = (-v\sin s\,,\, v\cos s\,,\, 1).$

That's means, the surface is a part of plane. Then, we find

$\mathbf{G} = (0,0,-1), \quad H = 0, \quad \Delta\mathbf{G} = 0, \quad \Delta\mathbf{x} = 0.$

Hence, $\mathbf{G}$ has pointwise $1-$type Gauss map of the first kind and M_1 is a minimal surface of $1-$type.

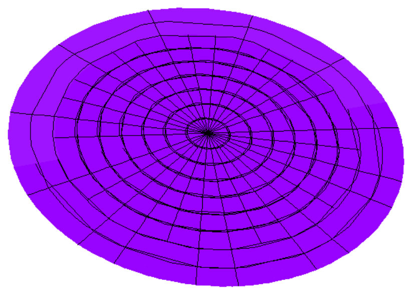

Fig. 1: M_1, $s \in [\frac{-3\pi}{2}, \frac{3\pi}{2}], v \in [-10, 13]$

4 Tangential Darboux developable ruled surfaces with $1-$type Gauss map

Let M_2 be the tangential Darboux developable ruled surface. Then, the parametrization of M_2 is written as follow:

$$M_2 : \mathbf{X}(s,v) = v\,\mathbf{n}(s) + \frac{\tau\,\mathbf{t}(s) + k\,\mathbf{b}(s)}{\sqrt{\tau^2 + k^2}}. \tag{21}$$

Then,

$$\mathbf{G} = -\frac{(v\,\eta^3 + \mu)(k\,\mathbf{b} + \tau\,\mathbf{t})}{\eta\,\sqrt{3v^2\,k^2\,\eta^2\,\tau^2 + v^2(k^6 + \tau^6) + 2v\,\mu\,\eta^3 + \mu^2}}, \tag{22}$$

where, $\eta^2 = \tau^2 + k^2 \neq 0$ and $\mu = \tau\,k' - k\,\tau'$. Therefore,

$$(g_{ij}) = diag\left(\frac{(v\eta^3 + \mu)^2}{\eta^4}\,,1\right),$$

$$g = \frac{(v\,\eta^3 + \mu)^2}{\eta^4}. \tag{23}$$

Thus, the formula of the Laplacian Δ of M_2 is given by

$$\begin{aligned}\Delta =& \frac{-1}{2g^2\,\eta^4}\Big(2g\,\eta^4\,\frac{\partial^2}{\partial\,s^2} + 2g\,(v\,\eta^3 + \mu)^2\,\frac{\partial^2}{\partial\,v^2} - \eta^4\,g_s\,\frac{\partial}{\partial\,s} \\ &+ (v\,\eta^3 + \mu)\,\big(4g\,\eta^3 - (v\,\eta^3 + \mu)\,g_v\big)\,\frac{\partial}{\partial\,v}\Big),\end{aligned} \tag{24}$$

where $g_s = \frac{\partial g}{\partial s}$, $g_v = \frac{\partial g}{\partial v}$ and are given by

$$g_s = \frac{2}{\eta^5}\,(\mu + v\,\eta^3)\,\big(\eta\,\mu' + \eta'\,(\eta^3 v - 2\mu)\big), \quad g_v = \frac{2(\mu + v\,\eta^3)}{\eta}.$$

Therefore,

$$H=\frac{\mu}{2\sqrt{3v^2\,k^2\,\eta^2\,\tau^2+v^2\,(k^6+\,\tau^6)+2v\,\mu\,\eta^3+\mu^2}}. \tag{25}$$

Consequently,

$$\Delta\,\mathbf{X}=\frac{\mu\,(k\,\mathbf{b}+\tau\,\mathbf{t})}{\eta\,(v\,\eta^3+\mu)}. \tag{26}$$

Now, we study some cases.

(a) For $\tau=0$

According to Eqs. (22)-(26), we get

$$\mathbf{G}=-\mathbf{b},\quad \Delta\mathbf{G}=0,\quad H=0,\quad \Delta\mathbf{X}=0.$$

Hence, $\mathbf{G}$ has pointwise $1-$type Gauss map of the first kind and $\mathbf{X}$ is $1-$type.

(b) For a space curve (helix) k **and** τ **are constants.**

In this case, we get

$$\mathbf{G}=-\frac{\eta^2\,(k\,\mathbf{b}+\tau\,\mathbf{t})}{\sqrt{k^6+3\,\eta^2\,k^2\,\tau^2+\tau^6}},$$

$$\Delta\mathbf{G}=0, H=0,\ \Delta\,\mathbf{X}=0.$$

This means, $\mathbf{G}$ and $\mathbf{X}$ are $1-$type also.

Theorem 2.*The finite type of tangential Darboux developable ruled surface M_2 in E^3 with plane curve or circular helix as a base curve is $1-$type and has pointwise $1-$type Gauss map of the first kind.*

5 Rectifying developable ruled surfaces

Let M_3 be the rectifying developable ruled surface. Then, the parametrization of M_3 is expressed as the following:

$$M_3:\mathbf{X}(s,v)=\alpha(s)+v\,\frac{\tau\,\mathbf{t}(s)+k\,\mathbf{b}(s)}{\sqrt{\tau^2+k^2}}. \tag{27}$$

Then,

$$\mathbf{G}=-\mathbf{n}. \tag{28}$$

Therefore,

$$(g_{ij})=\begin{pmatrix}1+(v\,\mu\,(-2k\,\eta+v\,\mu))/\eta^4 & \frac{\tau}{\eta}\\ \frac{\tau}{\eta} & 1\end{pmatrix}, \tag{29}$$

$$g=\frac{(k\,\eta-v\,\mu)^2}{\eta^4}, \tag{30}$$

where, $\eta^2=\tau^2+k^2\neq0$ and $\mu=\tau\,k'-k\,\tau'$. Then, the formula of the Laplacian Δ of M_3 is given by

$$\Delta=\frac{-1}{2g^2\,\eta^4}\Big(2g\,\eta^4\,\frac{\partial^2}{\partial s^2}+2g\,(\eta^4-2\eta\,k\,v\,\mu+v^2\,\mu^2)\,\frac{\partial^2}{\partial v^2}-\eta^3\,(\eta\,g_s-\tau\,g_v)\,\frac{\partial}{\partial s}+ \\ (-\eta^4\,g_v-2k\,\eta\,\mu\,(g-v\,g_v)+v\,(4g-v\,g_v)\,\mu^2+\eta^3\,\tau\,g_s)\,\frac{\partial}{\partial v}-4g\,\eta^3\,\tau\,\frac{\partial^2}{\partial s\,\partial v}\Big), \tag{31}$$

where

$$g_s = \frac{-1}{\eta^7}\big(2\eta^2\,\tau\,\tau'\,(\eta^3 - \eta\,\tau^2 - v\,k\,\mu) - 2\eta\,(\eta\,k - v\,\mu)\,(-v\,\eta^2\,\tau\,k'' + \eta\,\tau^2\,k' + 2v\,k\,\mu\,k' + v\,\eta^2\,k\,\tau'' + 2v\,\mu\,\tau\,\tau')\big),$$

$$g_v = \frac{-2\mu}{\eta^5}\,\big(k^3 - v\,\eta\,\tau\,k' + k\,(\tau^2 + v\,\eta\,\tau')\big).$$

Therefore,

$$H = \frac{\eta^3}{2(v\,\mu\, - k\,\eta)}. \tag{32}$$

Consequently,

$$\Delta\,\mathbf{X} = \frac{\eta^3\,\mathbf{n}}{v\,\mu\, - k\,\eta}. \tag{33}$$

Here, we study two cases:

(a) For α is a plane curve.

Putting $\tau = 0$ in the previous equations, we get

$$\mathbf{G} = -\mathbf{n}, \quad \Delta\,\mathbf{G} = k^2\,\mathbf{G} - k'\,\mathbf{t}. \tag{34}$$

This means, $\mathbf{G}$ has pointwise $1-$type Gauss map of the first kind if $k' = 0$, i.e, k is a constant. Then we conclude the following theorem:

Theorem 3. *The rectifying developable ruled surfaces M_3 has pointwise $1-$type Gauss map of the first kind with a base plane curve if and only if the base curve is a circle or straight line.*

Additionally, we get

$$H = \frac{-k}{2}, \quad \Delta\,\mathbf{X} = -k\,\mathbf{n}. \tag{35}$$

For $\mathbf{X}$ be finite type we get the following theorem:

Theorem 4. *The finite type ruled surface M_3 with a base plane curve has one of the following properties:*

(i)If $\lambda_1 = \lambda_2 = 0$, then M_3 is a part of plane of $1-$type.
(ii)If $\lambda_1 = \lambda_2 \neq 0$, then M_3 is a null $2-$type.
(iii)If $\lambda_1 \neq \lambda_2$, then M_3 is less or equal to null $3-$type.

Proof. In view of Eqs. (4) and (5) we can write (27) as a spectral decomposition in vector form as the following:

$$\mathbf{X}_1 = (\alpha_1, 0, 0), \quad \mathbf{X}_2 = (0, \alpha_2, 0), \quad \mathbf{X}_3 = \big(0, 0, \frac{v}{k}(\alpha_1'\alpha_2'' - \alpha_1''\alpha_2')\big). \tag{36}$$

After some computations one can get

$$\lambda_i = -\frac{\alpha_i''}{\alpha_i}; \quad i = 1, 2, \quad \lambda_3 = 0.$$

Since, λ_i must be constant, there are three possibilities of λ_i which are stated in the theorem.

(b) For a space curve k and τ are constants.

Substituting τ and k in Eqs. (28)-(31), we get

$$\mathbf{G} = -\mathbf{n}, \quad \Delta\,\mathbf{G} = \frac{\eta^4}{k^2}\,\mathbf{G}, \quad k \neq 0. \tag{37}$$

Consequently, $\mathbf{G}$ is pointwise $1-$type of the first kind.

Theorem 5. *The rectifying developable ruled surface M_3 has pointwise $1-$type Gauss map of the first kind if the base plane curve is a circular helix.*

Example 2. Let α be a helix curve which has the parametrization as follow:

$$\alpha = \frac{1}{\sqrt{13}}(3\cos s\,,\,3\sin s\,,\,2s). \tag{38}$$

Then, $k = \frac{3}{\sqrt{13}}$ and $\tau = \frac{2}{\sqrt{13}}$. Consequently, the rectifying developable ruled surface is parameterized by

$$\mathbf{X} = \frac{1}{\sqrt{13}}(3\cos s,\,3\sin s,\,2s+v\sqrt{13}).$$

Hence,

$$\mathbf{G} = (\cos s,\,\sin s,\,0), \quad \Delta\,\mathbf{G} = \frac{13}{9}\mathbf{G}, \quad H = -\frac{\sqrt{13}}{6}.$$

Therefore, $\mathbf{G}$ is pointwise $1-$type of the first kind. Then,

$$\Delta\mathbf{X} = \frac{\sqrt{13}}{3}(\cos s,\,\sin s,\,0).$$

Using Eq. (5), we get

$$\frac{1}{3\sqrt{13}}\big((13-9\lambda_1)\cos s,(13-9\lambda_2)\sin s,-\lambda_3\,(6s+3v\sqrt{13})\big) = 0.$$

Solving this equation, we get $\lambda_1 = \lambda_2 = \frac{13}{9}$, and $\lambda_3 = 0$. Thus, the surface M_3 is a null $2-$type. Look Figure 2.

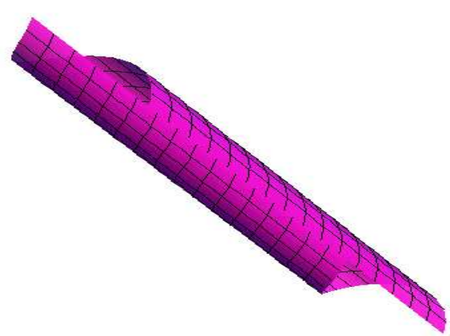

Fig. 2: $M_9,\, s \in [0,2\pi], v \in [-5,5]$

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