Applied Mathematics & Information Sciences Letters
*An International Journal*

http://dx.doi.org/10.18576/amisl/040302

# Integral Inequalities for Geometrically log-Preinvex Functions

*Muhammad Aslam Noor**, *Sundas Khan and Khalida Inayat Noor*

Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.

Received: 1 Apr. 2016, Revised: 15 Apr. 2016, Accepted: 18 Apr. 2016
Published online: 1 Sep. 2016

**Abstract:** In this paper, we introduce and investigate a new class of preinvex functions, which is called geometrically log-preinvex functions. Some new Hermite-Hadamard type inequalities are obtained. The idea and technique of this paper may stimulate further research in this field.

**Keywords:** Convex functions,preinvex functions, Hermite-Hadamard type inequalities, geometrically preinvex functions, geometrically log-preinvex functions.

## 1 Introduction

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson [1]. Hansons initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in optimization and other branches of pure and applied sciences. Weir and Mond [15], Jeyakumar and Mond [3] have studied the basic properties of the preinvex functions. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets. Inspired and motivated by the research going on in this field, we introduce and investigate a new class of convex functions, which is called the geometrically log-preinvex functions. For some recent investigations, see [9, 18, 19]. We establish the relationship between these classes and derive some new results. As special cases, one can obtain some new and correct versions of known results. Results obtained in this paper present a refinement and improvement of previously known results.

**Definition 1.**[1] *The set $K$ is said to be a convex set, if*

$$ta+(1-t)b \in K, \qquad \forall a,b \in K, t \in [0,1].$$

**Definition 2.**[2] *Let $K$ be the convex set. Then the function $f$ defined on $K$ is said to be convex function, if*

$$f(ta+(1-t)b) \le tf(a)+(1-t)f(b), \forall a,b \in K, t \in [0,1].$$

**Definition 3.**[4] *The set $K_\eta$ is said to be invex with respect to $\eta(.,.)$, if, there exists a bifunction $\eta(.,.)$ such that*

$$a+t\eta(b,a) \in K_\eta, \qquad \forall a,b \in K_\eta, t \in [0,1].$$

*The invex set is also called a $\eta-connected$ set.*

We would like to mention that Definition of an invex set has a clear geometric interpretation. If we demand that $a$ should be an end point of the path for every pair of points $a,b \in K$, then $\eta(b,a) = b-a$, and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(.,.)$, but the converse is not necessarily true, see [15, 16].

For the sake of simplicity, we always assume that $I_\eta = [a, a+\eta(b,a)]$, unless otherwise specified.

**Definition 4.**[9] *A function $f$ on the invex set $K_\eta \times K_\eta \to R$ is said to be preinvex with respect to $\eta(.,.)$, if there exists a bifunction $\eta(.,.)$ such that*

$$f(a+t\eta(b,a)) \le (1-t)f(a)+tf(b), \forall a,b \in K_\eta, t \in [0,1].$$

If $\eta(b,a) = b-a$, then invex set reduces to convex set, see [9]. We now recall the concepts of the geometrically convex sets and geometrically convex functions, see [9] and the references their.

**Definition 5.**[21] *Let $I_\eta \subset (0,\infty)$. The set $I_\eta$ is said to be geometrically invex set, if*

$$a^t(a+\eta(b,a))^{(1-t)} \in I_\eta, \forall a, a+\eta(b,a) \in I_\eta, t \in [0,1].$$

* Corresponding author e-mail: noormaslam@hotmail.com

**Definition 6.**[21] *A function $f: I_\eta = [a, a+\eta(b,a)]$ is said to be geometrically preinvex on $I_\eta$, if inequality.*

$$f(a^t(a+\eta(b,a))^{(1-t)}) \\ \leq tf(a)+(1-t)f(b), \forall a, a+\eta(b,a) \in I_\eta.$$

If we take *log* on both sides, we have

$$lnf(a^t(a+\eta(b,a))^{(1-t)}) \\ \leq ln([f(a)]^{1-t}[f(b)]^t), \forall a, a+\eta(b,a) \in I_\eta. \\ \leq (1-t)lnf(a)+tlnf(b)$$

**Definition 7.** *A function $f: I_\eta = [a, a+\eta(b,a)]$ is said to be geometrically log-preinvex on $I_\eta$, if inequality.*

$$\begin{aligned} f(a^t(a+\eta(b,a))^{(1-t)}) &\leq [f(a)]^{1-t}[f(b)]^t, \\ &\leq (1-t)f(a)+tf(b) \\ &\leq max\{f(a), f(b)\} \end{aligned}$$

Thus geometrically log-preinvex reduces to GA-preinvex function and then it reduces to quasi log-preinvex function, but converse is not true. We remark that if $\eta(b,a) = b-a$, then $I_\eta$ reduces to $I$ and consequently definition (1.7) is exactly the definition (1.4). This shows geometrically-convex functions are geometrically-preinvex functions, but the converse is not true.

We define the logrithemic mean $L(a,b)$ of two positive numbers $a, b$ by

$$Ł(a,b) = \begin{cases} \frac{b-a}{lnb-lna}, & \text{if} \quad b \neq a \\ b, & \text{if} \quad b = a \end{cases} \tag{1}$$

We also define

$$M = \frac{g(b)-g(a)}{(lna-ln(a+\eta(b,a)))(lng(a+\eta(b,a))-lng(a))}$$

$$N = \frac{f(b)-f(a)}{(lna-ln(a+\eta(b,a)))(lnf(a+\eta(b,a))-lnf(a))}$$

## 2 Main results

**Theorem 1.** *Let $f: I_\eta \to (0,\infty)$ be an increasing and a geometrically log-preinvex function on $I_\eta$ and $a, a+\eta(b,a) \in I_\eta$ with $\eta(b,a) > 0$. Then following inequality holds:*

$$\frac{8(f(b)-f(a))}{(lna-ln(a+\eta(b,a)))(lnf(b)-lnf(a))}\int_a^{a+\eta(b,a)}\frac{f(x)}{x}dx \\ \leq \frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)}\frac{f^4(x)}{x}dx \\ +\left(\frac{f^2(a)+f^2(b)}{2}\right)\left(\frac{f(a)+f(b)}{2}\right)\left(\frac{f(a)-f(b)}{lnf(a)-lnf(b)}\right)+8.$$

*Proof.* Since $f$ is geometrically log-preinvex function on $I_\eta$, thus

$$f(a^{1-t}(a+\eta(b,a))^t) \leq [f(b)]^t[f(a)]^{1-t} \\ \forall a, a+\eta(b,a) \in I_\eta, t \in [0,1].$$

for all $a, a+\eta(b,a) \in I_\eta$ and $t \in [0,1]$. Using the inequality $8xy \leq x^4+y^4+8$, we have that

$$8f(a^{1-t}(a+\eta(b,a))^t)[f(a)]^{1-t}[f(b)]^t \\ \leq f^4(a^{1-t}(a+\eta(b,a))^t)+[f(a)]^{4t}[f(b)]^{4(1-t)}+8.$$

Integrating the above inequality over $t$ on $[0,1]$, we get the inequality

$$8\int_0^1 f(a^{1-t}(a+\eta(b,a))^t)(f(a)^{1-t})(f(b))^t dt \\ \leq \int_0^1 f^4(a^{1-t}(a+\eta(b,a))^t)dt \\ +\int_0^1 [f(a)]^{4t}[f(b)]^{4(1-t)}dt+8.$$

Since $f$ is increasing and continuous, we have

$$8\int_0^1 f(a^{1-t}(a+\eta(b,a))^t)dt\int_0^1 [f(a)]^{1-t}[f(b)]^t dt \\ \leq \int_0^1 f^4(a^{1-t}(a+\eta(b,a))^t)dt \\ +\int_0^1 [f(a)]^{4t}[f(b)]^{4(1-t)}dt+8.$$

Then, we obtain

$$\frac{8(f(b)-f(a))}{(lna-ln(a+\eta(b,a)))(lnf(b)-lnf(a))}\int_a^{a+\eta(b,a)}\frac{f(x)}{x}dx \\ \leq \frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)}\frac{f^4(x)}{x}dx \\ +\left(\frac{f^2(a)+f^2(b)}{2}\right)\left(\frac{f(a)+f(b)}{2}\right)\left(\frac{f(a)-f(b)}{lnf(a)-lnf(b)}\right)+8.$$

This completes the proof. □

**Theorem 2.** *Let $f, g: I_\eta \to (0,\infty)$ be an increasing and geometrically log-preinvex functions on $I_\eta$ and $a, a+\eta(b,a) \in I_\eta$ with $\eta(b,a) > 0$. Then*

$$\frac{L(g(a),g(b))}{(lna-ln(a+\eta(b,a)))}\int_a^{a+\eta(b,a)}\frac{f(x)}{x}dx \\ +\frac{L(f(a),f(b))}{(lna-ln(a+\eta(b,a)))}\int_a^{a+\eta(b,a)}\frac{g(x)}{x}dx \\ \leq \frac{1}{(lna-ln(a+\eta(b,a)))}\int_a^{a+\eta(b,a)}\frac{f(x)g(x)}{x^2}dx \\ +\frac{f(b)g(b)-f(a)g(a))}{ln(f(b)g(b))-ln(f(a)g(a))}$$

*Proof.* Since $f$ and $g$ are geometrically log-preinvex functions, we have that

$$f(a^{1-t}(a+\eta(b,a))^t) \le [f(a)]^{1-t}[f(b)]^t$$
$$g(a^{1-t}(a+\eta(b,a))^t) \le [g(a)]^{1-t}[g(b)]^t$$

Now, using the elementary inequality, $ac+bd > bc+ad$, $(a,b,c,d) \in R$ and $(a<b, c<d)$, we get the inequality

$$f(a^{1-t}(a+\eta(b,a))^t)[g(a)]^{1-t}[g(b)]^t$$
$$+g(a^{1-t}(a+\eta(b,a))^t)[f(a)]^t[f(b)]^{1-t} > 0$$
$$f(a^{1-t}(a+\eta(b,a))^t)g(a^{1-t}(a+\eta(b,a))^t)$$
$$+[g(a)]^{1-t}[g(b)]^t[f(a)]^t[f(b)]^{1-t} > 0.$$

Take the integral of ithe above nequality over t on $[0,1]$,we have

$$I_1 = \int_0^1 f(a^{1-t}(a+\eta(b,a))^t)[g(a)]^{1-t}[g(b)]^t dt$$

$$+\int_0^1 g(a^{1-t}(a+\eta(b,a))^t)[f(a)]^{1-t}[f(b)]^t dt$$

$$I_2 = \int_0^1 f(a^{1-t}(a+\eta(b,a))^t)g(a^{1-t}(a+\eta(b,a))^t)dt$$

$$+\int_0^1 [f(a)g(a)]^{1-t}[g(b)f(a)]^t dt$$

Integrating $I_1$,

$$I_1 = \int_0^1 f(a^{1-t}(a+\eta(b,a))^t)[g(a)]^{1-t}[g(b)]^t dt$$

$$+\int_0^1 g(a^{1-t}(a+\eta(b,a))^t)[f(a)]^{1-t}[f(b)]^t dt$$

$$\ge \int_0^1 f(a^{1-t}(a+\eta(b,a))^t)dt \int_0^1 [g(a)]^{1-t}[g(b)]^t dt$$

$$+\int_0^1 g(a^{1-t}(a+\eta(b,a))^t)dt \int_0^1 [f(a)]^{1-t}[f(b)]^t dt$$

$$= \frac{g(b)-g(a)}{(lna-ln(a+\eta(b,a)))(lng(a+\eta(b,a))-lng(a))}$$
$$\times \int_a^{a+\eta(b,a)} \frac{f(x)}{x}dx$$
$$+\frac{f(b)-f(a)}{(lna-ln(a+\eta(b,a)))(lnf(a+\eta(b,a))-lnf(a))}$$
$$\times \int_a^{a+\eta(b,a)} \frac{g(x)}{x}dx$$

Now integrating $I_2$,

$$I_2 = \int_0^1 f(a^{1-t}(a+\eta(b,a))^t)g(a^{1-t}(a+\eta(b,a))^t)dt$$

$$+\int_0^1 [f(a)g(a)]^{1-t}[g(b)f(a)]^t dt$$

$$= \frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)} \frac{f(x)g(x)}{x^2}dx$$

$$+f(a)g(a)\int_0^1 \left(\frac{f(b)g(b)}{f(a)g(a)}\right)^t dt$$

$$= \frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)} \frac{f(x)g(x)}{x^2}dx$$

$$+\frac{f(b)g(b)-f(a)g(a)}{(lnf(b)g(b)-lnf(a)g(a))}$$
$$= \frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)} \frac{f(x)g(x)}{x^2}dx$$
$$+L(f(a)g(a), f(b)g(b)))$$

$$= \frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)} \frac{f(x)g(x)}{x^2}dx$$
$$+\frac{f(a)g(a)+f(b)g(b)}{2}$$

Since $I_1 \le I_2$

$$M\int_a^{a+\eta(b,a)} \frac{f(x)}{x}dx + N\int_a^{a+\eta(b,a)} \frac{g(x)}{x}dx$$

$$\le \frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)} \frac{f(x)g(x)}{x^2}dx$$
$$+\frac{f(a)g(a)+f(b)g(b)}{2}.$$

**Theorem 3.** *Let $f : I_\eta \to (0,\infty)$ be an increasing and geometrically log-preinvex function on $I_\eta$ and $a, a+\eta(b,a) \in I_\eta$ with $\eta(b,a) > 0$. Then*

$$\frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)} \frac{f(x)}{x}dx$$
$$+\frac{(f(b)+f(a))(f(b)-f(a))}{2(lnf(b)-lnf(a))}$$
$$+\frac{f^2(a)+f(a)f(b)+f^2(b)}{3}$$

$$\geq \frac{f(b)-f(a)}{lnf(b)-lnf(a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x}dx$$
$$+\frac{f^2(a)}{lnf(b)-lnf(a)}$$
$$+\frac{f(b)f(a)-f^2(b)}{lnf^2(a)-lnf^2(b)}+\frac{f^2(b)}{lnf(b)-lnf(a)}$$
$$-\frac{f^2(b)-f(b)f(a)}{lnf^2(b)-lnf^2(a)}$$
$$+\frac{f(a)}{(ln(a+\eta(b,a))-lna)^2}$$
$$\times\int_a^{a+\eta(b,a)}\left(\frac{ln(a+\eta(b,a))-lnx}{x}\right)^2 f(x)dx$$
$$+\frac{f(b)}{(ln(a+\eta(b,a))-lna)^2}$$
$$\times\int_a^{a+\eta(b,a)}\frac{lnx-lna}{x}f(x)dx$$

*Proof.*Since $f$ is geometrically log-preinvex function on $I_\eta$, we have that

$$f(a^{1-t}(a+\eta(b,a))^t)\leq[f(a)]^{1-t}[f(b)]^t\leq(1-t)f(a)+tf(b)$$

for all $a,a+\eta(b,a)\in I_\eta$, and $t\in[0,1]$. Using the elementary inequality, $xy+yz+zx\leq x^2+y^2+z^2$, $(x,y,z\in R)$, we observe that

$$f^2(a^{1-t}(a+\eta(b,a))^t)$$
$$+[f(a)]^{2(1-t)}[f(b)]^{2t}+t^2f^2(b)$$
$$+(1-t)^2f^2(a)+2t(1-t)f(a)f(b)$$
$$\geq f(a^{1-t}(a+\eta(b,a))^t)[f(a)]^{1-t}[f(b)]^t$$
$$+[f(a)]^{1-t}[f(b)]^t((1-t)f(a)+tf(b))$$
$$+f(a^{1-t}(a+\eta(b,a))^t)((1-t)f(a)+tf(b))$$

Integrating the above inequality over $t$ on $[0,1]$, we deduce that

$$I_1=\int_0^1 f^2(a)(a^{1-t}(a+\eta(b,a))^t)dt$$
$$+\int_0^1[f(a)]^{2(1-t)}[f(b)]^{2t}dt$$
$$+\int_0^1 t^2f^2(a)dt+\int_0^1(1-t)^2f^2(a)dt$$
$$+\int_0^1 2t(1-t)f(a)f(b)dt$$
$$I_1=\frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)}\frac{f(x)}{x}dx$$
$$+\frac{(f(b)+f(a))(f(b)-f(a))}{2(lnf(b)-lnf(a))}$$
$$+\frac{f^2(a)+f(a)f(b)+f^2(b)}{3}.$$

Take $I_2$

$$I_2=\int_0^1 f(a^{1-t}(a+\eta(b,a))^t)[f(a)]^{1-t}[f(b)]^t dt$$
$$+\int_0^1[f(a)]^{1-t}[f(b)]^t t(1-t)f(a)f(b)$$
$$+\int_0^1 f(a^{1-t}(a+\eta(b,a))^t)((1-t)f(a)+tf(b))dt$$
$$I_2=\frac{f(b)-f(a)}{lnf(b)-lnf(a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x}dx+\frac{f^2(a)}{lnf(b)-lnf(a)}$$
$$+\frac{f(b)f(a)-f^2(a)}{lnf^2(a)-lnf^2(b)}+\frac{f^2(b)}{lnf(b)-lnf(a)}$$
$$-\frac{f^2(b)-f(b)f(a)}{lnf^2(b)-lnf^2(a)}+\frac{f(a)}{(ln(a+\eta(b,a))-lna)^2}$$
$$\times\int_a^{a+\eta(b,a)}\left(\frac{ln(a+\eta(b,a))-lnx}{x}\right)^2 f(x)dx$$
$$+\frac{f(b)}{(ln(a+\eta(b,a))-lna)^2}\int_a^{a+\eta(b,a)}\frac{lnx-lna}{x}f(x)dx$$

$I_1\geq I_2$, implies that

$$\frac{1}{lna-ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)}\frac{f(x)}{x}dx$$
$$+\frac{(f(b)+f(a))(f(b)-f(a))}{2(lnf(b)-lnf(a))}$$
$$+\frac{f^2(a)+f(a)f(b)+f^2(b)}{3}$$
$$\geq\frac{f(b)-f(a)}{lnf(b)-lnf(a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x}dx$$
$$+\frac{f^2(a)}{lnf(b)-lnf(a)}$$
$$+\frac{f(b)f(a)-f^2(b)}{lnf^2(a)-lnf^2(b)}$$
$$+\frac{f^2(b)}{lnf(b)-lnf(a)}$$
$$-\frac{f^2(b)-f(b)f(a)}{lnf^2(b)-lnf^2(a)}$$
$$+\frac{f(a)}{(ln(a+\eta(b,a))-lna)^2}$$
$$\times\int_a^{a+\eta(b,a)}\left(\frac{ln(a+\eta(b,a))-lnx}{x}\right)^2 f(x)dx$$
$$+\frac{f(b)}{(ln(a+\eta(b,a))-lna)^2}$$
$$\times\int_a^{a+\eta(b,a)}\frac{lnx-lna}{x}f(x)dx.$$

**Theorem 4.***Let $f:[a,a+\eta(b,a)]\subset(0,\infty)\to(0,\infty)$ be a geometrically log-preinvex function on $[a,a+\eta(b,a)]$.*

*Then for any $p > 0$, we have*

$$\frac{1}{lna - ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)} x^{q-1} f^p(x)dx$$
$$\leq \frac{(a+\eta(b,a))f^p(b) - af^p(a)}{ln(a+\eta(b,a))f^p(b) - lnaf^p(a)}$$

*Proof.*Consider

$$f^p(a^{1-t}(a+\eta(b,a))^t) \leq f^p[(a)]^{1-t}[f^p(b)]^t$$

$$f^p(a^{1-t}(a+\eta(b,a))^t)(a^{1-t}(a+\eta(b,a))^t)^q$$

$$\leq [f^p(a)]^{1-t}[f^p(b)]^t(a^{1-t}(a+\eta(b,a))^t)$$

$$\leq [af^p(a)]^{1-t}[(a+\eta(b,a))f^p(b)]^t$$

Integrate from $0 \to 1$

$$\int_0^1 f^p(a^{1-t}(a+\eta(b,a))^t)(a^{1-t}(a+\eta(b,a))^t)^q dt$$

$$\leq \int_0^1 [af^p(a)]^{1-t}[(a+\eta(b,a))f^p(b)]^t$$

$$\leq af^p(a)\int_0^1 \left(\frac{(a+\eta(b,a))f^p(b)}{af^p(a)}\right)^t dt$$

$$= \frac{(a+\eta(b,a))f^p(b) - af^p(a)}{ln(a+\eta(b,a))f^p(b) - lnaf^p(a)}$$

Now take

$$\int_0^1 f^p(a^{1-t}(a+\eta(b,a))^t)(a^{1-t}(a+\eta(b,a))^t)^q dt$$

$$= \frac{1}{lna - ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)} x^{q-1} f^p(x)dx,$$

which implies

$$\frac{1}{lna - ln(a+\eta(b,a))}\int_a^{a+\eta(b,a)} x^{q-1} f^p(x)dx$$
$$\leq \frac{(a+\eta(b,a))f^p(b) - af^p(a)}{ln(a+\eta(b,a))f^p(b) - lnaf^p(a)}$$

**Theorem 5.***Let $[a, a+\eta(b,a)] \subset (0,\infty) \to (0,\infty)$ be a geometrically log-preinvex function on $[a, a+\eta(b,a)]$. Then, for any $p > 0$, we have*

$$f^{2p}(\sqrt{a(a+\eta(b,a))})$$
$$\leq \frac{1}{ln(a+\eta(b,a)) - lna}\int_a^{a+\eta(b,a)} \frac{f^p(x)f^p\left(\frac{a(a+\eta(b,a))}{x}\right)}{x}dx$$
$$\leq f^p(a)f^p(b)$$

*Proof.*To prove theorem, we observe that if $f$ is geometrically log-preinvex, then we have

$$f(\sqrt{a(a+\eta(b,a))})$$
$$\leq \sqrt{f(a^{1-t}(a+\eta(b,a))^t)f(a^t(a+\eta(b,a))^{1-t})}$$
$$\leq \sqrt{f(a)f(b)},$$

for all $t \in [0,1]$. If we take the power $2p > 0$ in above equation, we get

$$f^{2p}(\sqrt{a(a+\eta(b,a))})$$
$$\leq f^p(a^{1-t}(a+\eta(b,a))^t)f^p(a^t(a+\eta(b,a))^{1-t})$$
$$\leq f^p(a)f^p(b)$$

By multiplying with $(a^{1-t}(a+\eta(b,a))^t) > 0$ for $q \in R\backslash\{0\}$ and integrating over $t$ on $[0,1]$ we get

$$f^{2p}(\sqrt{a(a+\eta(b,a))})\int_0^1 (a^{1-t}(a+\eta(b,a))^t)^q dt$$
$$\leq \int_0^1 f^p(a^{1-t}(a+\eta(b,a))^t)f^p(a^t(a+\eta(b,a))^{1-t})$$
$$\times(a^{1-t}(a+\eta(b,a))^t)^q dt$$
$$\leq f^p(a)f^p(b)\int_0^1 (a^{1-t}(a+\eta(b,a))^t)^q dt$$

For $q \neq 0,1$, substitute values in above expression, we get

$$f^{2p}(\sqrt{a(a+\eta(b,a))})\frac{a^q - (a+\eta(b,a))^q}{qlna - qln(a+\eta(b,a))}$$
$$\leq \frac{1}{ln(a+\eta(b,a)) - lna}\int_a^{a+\eta(b,a)} f^p(x)$$
$$f^p\left(\frac{a(a+\eta(b,a))}{x}\right)x^{q-1}dx$$
$$\leq f^p(a)f^p(b)\frac{a^p - (a+\eta(b,a))^q}{qlna - qln(a+\eta(b,a))}$$

For $q = 1$, we have

$$f^{2p}(\sqrt{a(a+\eta(b,a))})\frac{a - (a+\eta(b,a))}{lna - ln(a+\eta(b,a))}$$
$$\leq \frac{1}{ln(a+\eta(b,a)) - lna}\int_a^{a+\eta(b,a)} f^p(x)f^p\left(\frac{a(a+\eta(b,a))}{x}\right)dx$$
$$\leq f^p(a)f^p(b)\frac{a - (a+\eta(b,a))}{lna - ln(a+\eta(b,a))}$$

which is equivalent to

$$f^{2p}(\sqrt{a(a+\eta(b,a))})$$
$$\leq \frac{1}{-\eta(b,a)}\int_a^{a+\eta(b,a)} f^p(x)f^p\left(\frac{a(a+\eta(b,a))}{x}\right)dx$$
$$\leq f^p(a)f^p(b)$$

Using the fact that $\eta(b,a) = -\eta(b,a)$ (skew-symmetric), we get

$$\begin{aligned} & f^{2p}(\sqrt{a(a+\eta(b,a))}) \\ \leq & \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f^p(x) f^p\left(\frac{a(a+\eta(b,a))}{x}\right)dx \\ \leq & f^p(a)f^p(b) \end{aligned}$$

If $q = 0$, then

$$\begin{aligned} & f^{2p}(\sqrt{a(a+\eta(b,a))}) \\ \leq & \int_0^1 f^p(a^{1-t}(a+\eta(b,a))^t) f^p(a^t(a+\eta(b,a))^{1-t})dt \\ \leq & f^p(a)f^p(b), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & f^{2p}(\sqrt{a(a+\eta(b,a))}) \\ \leq & \frac{1}{ln(a+\eta(b,a)) - lna}\int_a^{a+\eta(b,a)} \frac{f^p(x) f^p\left(\frac{a(a+\eta(b,a))}{x}\right)}{x}dx \\ \leq & f^p(a)f^p(b) \end{aligned}$$

**Theorem 6.***Let $g : [a, a+\eta(b,a)] \subset (0,\infty) \to R$ be a geometrically log-preinvex function on $[a, a+\eta(b,a)]$. Then, for any $t \in [0,1]$,we have*

$$\begin{aligned} & f(\sqrt{a(a+\eta(b,a))}) \\ \leq & ([f(a^{\frac{1-t}{2}}(a+\eta(b,a))^{\frac{1+t}{2}})]^{1-t}[f(a^{\frac{2-t}{2}}(a+\eta(b,a))^{\frac{t}{2}})]^t) \\ \leq & \sqrt{f(a)f(b)} \end{aligned}$$

*Proof.*Consider

$$\begin{aligned} & g(\sqrt{a(a+\eta(b,a))}) \\ \leq & (1-t)g(a^{\frac{1-t}{2}}(a+\eta(b,a))^{\frac{1+t}{2}}) + tg(a^{\frac{2-t}{2}}(a+\eta(b,a))^{\frac{t}{2}}) \\ \leq & \frac{1}{ln(a+\eta(b,a)) - lna}\int_a^{a+\eta(b,a)} \frac{f(x)}{x}dx \\ \leq & \frac{1}{2}[g(a^{1-t}(a+\eta(b,a))^t) + (1-t)g(a+\eta(b,a)) + tg(a)] \\ \leq & \frac{g(a)+g(b)}{2} \end{aligned}$$

If $f$ is geometrically log-preinvex function, then, for $g = lnf$, we have

$$\begin{aligned} & lnf(\sqrt{a(a+\eta(b,a))}) \\ \leq & (1-t)lnf(a^{\frac{1-t}{2}}(a+\eta(b,a))^{\frac{1+t}{2}}) \\ & +tlnf(a^{\frac{2-t}{2}}(a+\eta(b,a))^{\frac{t}{2}}) \\ \leq & \frac{1}{ln(a+\eta(b,a)) - lna}\int_a^{a+\eta(b,a)} \frac{lnf(x)}{x}dx \\ \leq & \frac{1}{2}[lnf(a^{1-t}(a+\eta(b,a))^t) \\ & +(1-t)lnf(a+\eta(b,a)) + tlnf(a)] \\ \leq & \frac{lnf(a)+lnf(b)}{2} \end{aligned}$$

which is equivalent to

$$\begin{aligned} & lnf(\sqrt{a(a+\eta(b,a))}) \\ \leq & ln([f(a^{\frac{1-t}{2}})(a+\eta(b,a))^{\frac{1+t}{2}}]^{1-t}[f(a^{\frac{2-t}{2}}(a+\eta(b,a))^{\frac{t}{2}})]^t) \\ \leq & \frac{1}{ln(a+\eta(b,a)) - lna}\int_a^{a+\eta(b,a)} \frac{lnf(x)}{x}dx \\ \leq & ln(\sqrt{f(a^{1-t})(a+\eta(b,a))^t}[f(a+\eta(b,a))]^{1-t}[f(a)]^t) \\ \leq & ln(\sqrt{f(a)f(b)}) \end{aligned}$$

By taking exponential, we get the desired result

$$\begin{aligned} & f(\sqrt{a(a+\eta(b,a))}) \\ \leq & ([f(a^{\frac{1-t}{2}}(a+\eta(b,a))^{\frac{1+t}{2}})]^{1-t}[f(a^{\frac{2-t}{2}}(a+\eta(b,a))^{\frac{t}{2}})]^t) \\ \leq & \sqrt{f(a)f(b)} \end{aligned}$$

**Theorem 7.***Let $g : [a, a+\eta(b,a)] \subset (0,\infty) \to R$ be a geometrically log-preinvex function on $[a, a+\eta(b,a)]$. Then*

$$\begin{aligned} & \sqrt{f(a)f(a+\eta(b,a))} \\ \leq & exp\left(\frac{1}{(ln(a+\eta(b,a)) - lna)}\int_a^{a+\eta(b,a)} \frac{lnf(x)}{x}dx\right. \\ & \left.\left(\frac{a+\eta(b,a)}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(a+\eta(b,a))(a+\eta(b,a))}{f(a+\eta(b,a))} - \frac{f'_+(a)a}{f(a)}\right]}\right) \end{aligned}$$

*Proof.*Consider

$$\begin{aligned} 0 \leq & \frac{g(a)+g(a+\eta(b,a))}{2} - \frac{1}{ln(a+\eta(b,a)) - lna} \\ & \times \int_a^{a+\eta(b,a)} \frac{g(x)}{x}dx \\ \leq & \frac{1}{8}[g'_-(a+\eta(b,a))(a+\eta(b,a)) - g'_+(a)a] \\ & \times(ln(a+\eta(b,a)) - lna), \end{aligned}$$

and

$$\begin{aligned} 0 \leq & \frac{1}{ln(a+\eta(b,a)) - lna}\int_a^{a+\eta(b,a)} \frac{g(x)}{x}dx \\ & -g(\sqrt{a(a+\eta(b,a))}) \\ \leq & \frac{1}{8}[g'_-(a+\eta(b,a))(a+\eta(b,a)) \\ & -g'_+(a)a](ln(a+\eta(b,a)) - lna) \end{aligned}$$

If we take $g = lnf$, then we get

$$\frac{lnf(a)+lnf(a+\eta(b,a))}{2} - \frac{1}{(ln(a+\eta(b,a))-lna)}\int_a^{a+\eta(b,a)}\frac{lnf(x)}{x}dx \leq \frac{1}{8}\left[\frac{f'_-(a+\eta(b,a))(a+\eta(b,a))}{f(a+\eta(b,a))} - \frac{f'_+(a)a}{f(a)}\right](ln(a+\eta(b,a))-lna),$$

which is equivalent to

$$\frac{lnf(a)+lnf(a+\eta(b,a))}{2} \leq \frac{1}{ln(a+\eta(b,a))-lna}\int_a^{a+\eta(b,a)}\frac{lnf(x)}{x}dx + \frac{1}{8}\left[\frac{f'_-(a+\eta(b,a))}{f(a+\eta(b,a))-\frac{f'_+(a)a}{f(a)}}\right] ln\left(\frac{a+\eta(b,a)}{a}\right),$$

or to

$$ln(\sqrt{f(a)f(a+\eta(b,a))}) \leq ln\left(exp\left(\frac{1}{ln(a+\eta(b,a))-lna}\int_a^{a+\eta(b,a)}\frac{lnf(x)}{x}dx\right)\right)\left(\frac{a+\eta(b,a)}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(a+\eta(b,a))(a+\eta(b,a))}{f(a+\eta(b,a))}-\frac{f'_+(a)a}{f(a)}\right]}.$$

Taking the exponential of above equation

$$\sqrt{f(a)f(a+\eta(b,a))} \leq exp\left(\frac{1}{(ln(a+\eta(b,a))-lna)}\int_a^{a+\eta(b,a)}\frac{lnf(x)}{x}dx \times \left(\frac{a+\eta(b,a)}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(a+\eta(b,a))(a+\eta(b,a))}{f(a+\eta(b,a))}-\frac{f'_+(a)a}{f(a)}\right]}\right).$$

## Acknowledgement

The authors are grateful to Dr. S. M. Junaid Zaidi(H.I., S.I.), Rector, COMSATS Institute of Information Technology, Pakistan for providing the excellent research facilities.

## References

[1] M.A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl., 80 (1981), 545550.
[2] C.P. Niculescu, Convexity according to the geometric mean, Math. Inequal. Appl. 3, 155-167, (2000).
[3] V. Jeyakumar and B. Mond, On generalized convex mathematical programming, J. Austral. Math. Society Ser. B., 34 (1992), 4353.
[4] S.S. Dragomir, R. P. Agarwal, P. Cerone. On Simpson Inequality and Applications, J. Inequal. Appl. 5, 533-579, (2000).
[5] S.S. Dragomir, C. E. M. Pearce. Selected topics on Hermite-Hadamard inequalities and applications. Victoria University, Australia, (2000).
[6] S. S. Dragomir, Inequalities of Hermite-Hadamard type for GA-convex functions, Preprint RGMIA Res. Coll., 18, (2015).
[7] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications. Springer-Verlag, New York, (2006).
[8] M.Matloka, On some Hadamard-type inequalities for $(h_1,h_2)$-preinvex functions on the co-ordinates, J.Inequal.Appl., 222-227.
[9] Mevlut Tunc, Some integral inequalities for Logrithemically convex functions, Journal of Egyptian Mathematical Society, 22, 177-181, (2014).
[10] S.S. Dragomir, B. Mond, Integral inequalities of Hadamard type for Log-convex functions, Demonstratio Math,31, 354-364, (1998).
[11] S.S. Dragomir, Some remarks on Hadamard's Inequalities for convex functions, Extracta Math, 9, 88-94, (1994).
[12] Iscan, I. Hermite-Hadamard and Simpson like inequalities for differentiable harmonically convex functions, J. Math, 10, Article ID 346305, (2014).
[13] S.S. Dragomir, Two functions in connection to Hadamard's inequalities , J. Math. Anal. Appl, 167, 49-56, (1992).
[14] S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.
[15] Weir, T.; Mond, B.: Preinvex functions in multiple objective optimization. J. Math. Anal. Appl. 136, 2938 (1998).
[16] C.P. Niculescu, Convexity according to the geometric mean, Math. Inequal. Appl., 3, 155-167, (2000).
[17] M. Tunc, On Hadamard type inequalities for s-geometrically convex functions, Preprint, RGMIA Research Report Collection, vol.15, 6 pages, (2012).
[18] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, Journal of Inequalities and Applications, Volume 2010, Article ID 507560, 11 pages.
[19] M. A. Latif, New Hermite-Hadamard type inequalities for GA-convex functions with application, Analysis, 34, 379-389, (2014).
[20] M. A. Noor, Hermite-Hadamard integral inequalities for log-convex functions, J.Inequal. Pure Appl.Math, 8, 1-14, (2007)
[21] M. A. Noor, K. I. Noor and Sundas.Khan. Some Intrgral Inequalities for Geometrically Preinvex functions, Proceeding of Jangjeon Math. Soc,(2016).

**Muhammad Aslam Noor** earned his PhD degree from Brunel University, London, UK (1975) in the field of Applied Mathematics(Numerical Analysis and Optimization). He has vast experience of teaching and research at university levels in various countries including Pakistan, Iran, Canada, Saudi Arabia and UAE. His field of interest and specialization is versatile in nature. It covers many areas of Mathematical and Engineering sciences such as Variational Inequalities, Operations Research and Numerical Analysis. He has been awarded by the President of Pakistan: President's Award for pride of performance in recognition of his contributions in the field of Mathematical Sciences. He was awarded HEC Best Research award in 2009. He is currently member of the Editorial Board of several reputed international journals of Mathematics and Engineering sciences. He has more than 850 research papers to his credit which were published in leading international journals. He is highly cited researcher in mathematics according to Thomson Reutor(2.15).

**Sundas Khan** is currently PhD Scholar at COMSATS Institute of information and technology, Islamabad, Pakistan. Her field of interest is Convex Analysis.

**Khalida Inayat Noor** is an Eminent Foreign Professor at CIIT, Islamabad. She obtained her PhD from Wales University (UK). She has a vast experience of teaching and research at university levels in various countries including Iran, Pakistan, Saudi Arabia, Canada and United Arab Emirates. She was awarded HEC best research award in 2009 and CIIT Medal for innovation in 2009. She has been awarded by the President of Pakistan: Presidents Award for pride of performance on August 14, 2010 for her outstanding contributions in mathematical sciences and other fields. Her field of interest and specialization is Complex analysis, Geometric function theory, Functional and Convex analysis. She introduced a new technique, now called as Noor Integral Operator which proved to be an innovation in the field of geometric function theory and has brought new dimensions in the realm of research in this area. She has been personally instrumental in establishing PhD/ MS programs at CIIT. Dr. Khalida Inayat Noor has supervised successfully several Ph.D students and MS/M.Phil students. She has been an invited speaker of number of conferences and has published more than 400 (Four hundred ) research articles in reputed international journals of mathematical and engineering sciences. She is member of educational boards of several international journals of mathematical and engineering sciences.