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A Note on the Solution Set of a Fractional Integro-Differential Inclusion

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Abstract: We study an initial value problem associated to a integro-differential inclusion of fractional order defined by a multifunction with nonconvex values and we obtain a qualitative property of its solution set.

Keywords: Fractional derivative, differential inclusion, retract.

1 Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order [1,2,3]. The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. For the last achievements on fractional calculus and fractional differential equations we refer the reader to [4].In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [5], allows to use Cauchy conditions which have physical meanings. Recently, several qualitative results for fractional integro-differential equations were obtained in [6,7,8,9].

This paper is concerned with integro-differential inclusions of fractional order

$$D_c^p x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \; ([0,\infty)), \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1}$$

where $p \in (1,2]$, D_c^p denotes Caputo's fractional derivative, $F : [0,\infty) \times \mathbf{R} \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ is a multifunction and $x_0, x_1 \in \mathbf{R}$. $V : C([0,\infty),\mathbf{R}) \to C([0,\infty),\mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t,s,x(s))ds$ with $k(.,.,.) : [0,\infty) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function.

In the present note we prove that the set of selections of the multifunction F that correspond to the solutions of problem (1) is a retract of $L^1_{loc}([0,\infty),\mathbf{R})$. Our main hypothesis is that the multifunction is Lipschitz with respect to the second and third variable and the proof uses a well known selection theorem due to Bressan and Colombo [10] which gives continuous selections for multifunctions that are lower semicontinuous and with decomposable values.

We point out that in the classical case of differential inclusions several qualitative properties of solutions exists in the literature [11,12,13]. On one hand, our result may be seen as a generalization of of Theorem 3.4 in [14] proved for problems defined on bounded intervals and on the other hand, Theorem 1 below is a generalization to fractional framework of the main theorem in [13].

2 Preliminaries

In what follows $I := [0,T]$, $T > 0$, $\mathscr{L}(I)$ is the σ-algebra of all Lebesgue measurable subsets of I and $(X, |.|)$ is a real separable Banach. $C(I,X)$ denotes the space of continuous functions $x : I \to X$ with the norm $|x|_C = \sup_{t\in I}|x(t)|$ and $L^1(I,X)$ denotes the space of integrable functions $x : I \to X$ with the norm $|x|_1 = \int_0^T |x(t)|dt$.

The distance between a point $x \in X$ and a subset $A \subset X$ is defined by $d(x,A) = \inf\{|x-a|; a \in A\}$ and Pompeiu-Hausdorff distance between the closed subsets $A, B \subset X$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, $d^*(A,B) = \sup\{d(a,B); a \in A\}$.

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$\mathscr{P}(X)$ denotes the family of all nonempty subsets of X and with $\mathscr{B}(X)$ the family of all Borel subsets of X. For $A \subset I$ with $\chi_A(.) : I \to \{0,1\}$ we describe the characteristic function of A. Finally, for any $A \subset X$ $cl(A)$ is its the closure.

By definition a subset $D \subset L^1(I,X)$ it is *decomposable* if for any $u, v \in D$ and any subset $A \in \mathscr{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$.

We use the notation $\mathscr{D}(I,X)$ for the family of all decomposable closed subsets of $L^1(I,X)$.

In the next two results (S,d) is a separable metric space. By definition a set-valued map $H : S \to \mathscr{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $G \subset X$, the subset $\{s \in S;\ H(s) \subset G\}$ is closed. The next two Lemmas are proved in [10].

Lemma 2.1. *Consider $F^* : I \times S \to \mathscr{P}(X)$ a set-valued map with closed values, $\mathscr{L}(I) \otimes \mathscr{B}(S)$-measurable and $F^*(t,.)$ is l.s.c. for any $t \in I$.*

Then the set-valued map $H : S \to \mathscr{D}(I,X)$ defined by

$$H(s) = \{f \in L^1(I,X); \quad f(t) \in F^*(t,s) \quad a.e.\ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $q : S \to L^1(I,X)$ such that

$$d(0,F^*(t,s)) \le q(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

Lemma 2.2. *Consider $F : S \to \mathscr{D}(I,X)$ be a l.s.c. set-valued map with closed decomposable values and let $\psi : S \to L^1(I,X)$, $\phi : S \to L^1(I,\mathbf{R})$ be continuous mappings such that the set-valued map $H : S \to \mathscr{D}(I,X)$ given by*

$$H(s) = cl\{f(.) \in F(s); \quad |f(t) - \psi(s)(t)| < \phi(s)(t) \quad a.e.\ (I)\}$$

has nonempty values.

Then H admits a continuous selection, i.e. there exists $h : S \to L^1(I,X)$ continuous with $h(s) \in H(s) \quad \forall s \in S$.

Definition 2.3. [1]. a) *The fractional integral of order $p > 0$* of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$I^p f(t) = \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} f(s) ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt.$

b) *Caputo's fractional derivative of order $p > 0$* of a function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$D_c^p f(t) = \frac{1}{\Gamma(n-p)} \int_0^t (t-s)^{-p+n-1} f^{(n)}(s) ds,$$

where $n = [p] + 1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

Definition 2.4. A mapping $x \in C([0,\infty),\mathbf{R})$ is said to be a solution of problem (1) if there exists $h \in L^1_{loc}([0,\infty),\mathbf{R})$ with $h(t) \in F(t,x(t),V(x)(t))$, $a.e.$ $[0,\infty)$ satisfying $D_c^p x(t) = h(t)$ $a.e.$ $[0,\infty)$ and $x(0) = x_0, x'(0) = x_1$.

In the next section we are concerned with the following set associated to problem (1).

$$\begin{gathered} \tilde{h}(t) = x_0 + t x_1 + \int_0^t \frac{(t-u)^{p-1}}{\Gamma(p)} h(u) du, \\ \mathscr{S}(x_0,x_1) = \{h \in L^1_{loc}([0,\infty),\mathbf{R});\ h(t) \in F(t,\tilde{h}(t),V(\tilde{h})(t)) \quad a.e.\ [0,\infty)\}. \end{gathered} \tag{2}$$

3 The Result

Hypothesis 3.1. i) *The set-valued map $F(.,.,.) : [0,\infty) \times \mathbf{R} \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ is $\mathscr{L}([0,\infty)) \otimes \mathscr{B}(\mathbf{R} \times \mathbf{R})$ measurable and has nonempty closed values.*

ii) *For almost all $t \in I$, the set-valued map $F(t,.,.)$ is $L(t)$-Lipschitz in the sense that there exists $L(.) \in L^1_{loc}([0,\infty),\mathbf{R}_+)$ with*

$$d_H(F(t,x_1,y_1),F(t,x_2,y_2)) \le L(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall\ x_1,x_2,y_1,y_2 \in \mathbf{R}.$$

iii) *There exists a locally integrable function $q(.) \in L^1_{loc}([0,\infty),\mathbf{R})$ such that*

$$d_H(\{0\},F(t,0,V(0)(t))) \le q(t) \quad a.e.\ ([0,\infty)).$$

iv) $k(.,.,.) : [0,\infty) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *is a function such that* $\forall x \in \mathbf{R}$, $(t,s) \to k(t,s,x)$ *is measurable.*
v) $|k(t,s,x) - k(t,s,y)| \le L(t)|x-y| \quad a.e.\ (t,s) \in [0,\infty) \times [0,\infty), \forall x,y \in \mathbf{R}.$

We use next the following notations

$$M(t) := L(t)(1 + \int_0^t L(u)du), \quad t \in I, \quad |I^p M| := \sup_{t\in[0,\infty)} |I^p M(t)|.$$

$$\tilde{h}(t) = x_0 + tx_1 + \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} h(s)ds, \quad u \in L^1(I,\mathbf{R}), \tag{3}$$

$$q_0(h)(t) = |h(t)| + q(t) + L(t)|(|\tilde{h}(t)| + \int_0^t L(s)|\tilde{h}(s)|ds), \quad t \in I \tag{4}$$

Let us note that

$$d(h(t), F(t,\tilde{h}(t), V(\tilde{h})(t)) \le q_0(h)(t) \quad a.e.\ (I) \tag{5}$$

and for any $u_1, u_2 \in L^1(I,\mathbf{R})$

$$|q_0(h_1) - q_0(h_2)|_1 \le (1 + |I^p M(T)|)|h_1 - h_2|_1;$$

therefore, the mapping $q_0 : L^1(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ is continuous.

Also define

$$\mathscr{S}_I(x_0,x_1) = \{h \in L^1(I,\mathbf{R}); \quad h(t) \in F(t,\tilde{h}(t), V(\tilde{h})(t)) \quad a.e.\ (I)\}.$$

$$I_k = [0,k], \quad k \ge 1, \quad |h|_{1,k} = \int_0^k |h(t)|dt, \quad h \in L^1(I_k,\mathbf{R}).$$

The proof of the next result may be find in [14].

Lemma 3.2. *Suppose that Hypothesis 3.1 is verified and consider* $\phi : L^1(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ *a continuous function with* $\phi(h) = h$ *for all* $h \in \mathscr{S}_I(x_0,x_1)$. *If* $h \in L^1(I,\mathbf{R})$, *we put*

$$\Psi(h) = \{h \in L^1(I,\mathbf{R}); \quad h(t) \in F(t, \widetilde{\phi(h)}(t), V(\widetilde{\phi(h)})(t)) \quad a.e.\ (I)\},$$

$$\Phi(h) = \begin{cases} \{h\} & \text{if } h \in \mathscr{T}_I(x_0,x_1), \\ \Psi(h) & \text{otherwise}. \end{cases}$$

Then the set-valued map $\Phi : L^1(I,\mathbf{R}) \to \mathscr{P}(L^1(I,\mathbf{R}))$ *is l.s.c. with nonempty closed and decomposable values.*

Theorem 3.3. *Assume that Hypothesis 3.1 is satisfied, there exists* $|I^p M| < 1$ *and* $x_0, x_1 \in \mathbf{R}$.
Then there exists $G : L^1_{loc}([0,\infty),\mathbf{R}) \to L^1_{loc}([0,\infty),\mathbf{R})$ *continuous with the properties*
(i) $G(h) \in \mathscr{S}(x_0,x_1), \quad \forall h \in L^1_{loc}([0,\infty),\mathbf{R}),$
(ii) $G(h) = h, \quad \forall h \in \mathscr{S}(x_0,x_1).$

Proof. The idea of the proof consists in the construction, for every $k \ge 1$, of a sequence of continuous functions $g^k : L^1(I_k,\mathbf{R}) \to L^1(I_k,\mathbf{R})$ satisfying the following conditions
(I) $g^k(h) = h, \quad \forall h \in \mathscr{S}_{I_k}(x_0,x_1)$
(II) $g^k(h) \in \mathscr{S}_{I_k}(x_0,x_1), \quad \forall h \in L^1(I_k,\mathbf{R})$
(III) $g^k(h)(t) = g^{k-1}(h|_{I_{k-1}})(t), \quad t \in I_{k-1}$
If this construction is realized, we introduce $G : L^1_{loc}([0,\infty),\mathbf{R}) \to L^1_{loc}([0,\infty),\mathbf{R})$ with

$$G(h)(t) = g^k(h|_{I_k})(t), \quad k \ge 1.$$

The continuity of $g^k(.)$ and (III) allow to deduce that $G(.)$ is continuous. Taking into account (II), for each $h \in L^1_{loc}([0,\infty),\mathbf{R})$, we get

$$G(h)|_{I_k}(t) = g^k(h|_{I_k})(t) \in \mathscr{T}_{I_k}(x_0,x_1), \quad \forall k \ge 1,$$

which shows that $G(h) \in \mathscr{T}(x_0,x_1)$.

Consider $\varepsilon > 0$ and $m \ge 0$. We define $\varepsilon_m = \frac{m+1}{m+2}\varepsilon$. If $h \in L^1(I_1,\mathbf{R})$ and $m \ge 0$ we put

$$q_0^1(h)(t) = |h(t)| + q(t) + L(t)(|\tilde{h}(t)| + \int_0^t L(s)|\tilde{h}(s)|ds), \ t \in I_1$$

and

$$q^1_{m+1}(h) = |I^pM|^m(\frac{1}{\Gamma(p)}|q^1_0(h)|_{1,1} + \varepsilon_{m+1}).$$

Since the map $q^1_0(.) = q_0(.)$ is continuous, we find that $q^1_m : L^1(I_1,\mathbf{R}) \to L^1(I_1,\mathbf{R})$ is also continuous.

Set $g^1_0(h) = h$. In what follows, we show that for any $m \geq 1$ there exists $g^1_m : L^1(I_1,\mathbf{R}) \to L^1(I_1,\mathbf{R})$ continuous with the properties

$$g^1_m(h) = h, \quad \forall h \in \mathscr{S}_{I_1}(x_0,x_1), \tag{a_1}$$

$$g^1_m(h)(t) \in F(t,\widetilde{g^1_{m-1}(h)}(t),V(\widetilde{g^1_{m-1}(h)})(t)) \quad a.e.\ (I_1), \tag{b_1}$$

$$|g^1_1(h)(t) - g^1_0(h)(t)| \leq q^1_0(h)(t) + \varepsilon_0 \quad a.e.\ (I_1), \tag{c_1}$$

$$|g^1_m(h)(t) - g^1_{m-1}(h)(t)| \leq M(t)q^1_{m-1}(h) \quad a.e.\ (I_1), \quad m \geq 2. \tag{d_1}$$

If $h \in L^1(I_1,\mathbf{R})$, we define

$$\Psi^1_1(h) = \{f \in L^1(I_1,\mathbf{R});\ \ f(t) \in F(t,\widetilde{h}(t),V(\widetilde{h}(t))(t))\ \ a.e.(I_1)\},$$

$$\Phi^1_1(h) = \begin{cases} \{h\} & \text{if } h \in \mathscr{S}_{I_1}(x_0,x_1), \\ \Psi^1_1(h) & \text{otherwise.} \end{cases}$$

We apply Lemma 3.2 (with $\phi(h) = h$) and we deduce that $\Phi^1_1 : L^1(I_1,\mathbf{R}) \to \mathscr{D}(I_1,\mathbf{R})$ is l. s. c. Using (5) we obtain that the set

$$H^1_1(h) = cl\{f \in \Phi^1_1(u); \quad |f(t) - h(t)| < q^1_0(h)(t) + \varepsilon_0 \quad a.e.\ (I_1)\}$$

is not empty for any $h \in L^1(I_1,\mathbf{R})$. We apply the Lemma 2.2 to obtain a selection g^1_1 of H^1_1 which is continuous and verifies (a_1)-(c_1).

Assume that $g^1_i(.),\ i = 1,\dots m$ satisfying (a_1)-(d_1) are already constructed. Therefore, from Hypothesis 1 and (b_1), (d_1) we infer

$$\begin{gathered} d(g^1_m(h)(t),F(t,\widetilde{g^1_m(h)}(t),V(\widetilde{g^1_m(h)})(t)) \leq L(t)(|\widetilde{g^1_{m-1}(h)}(t) - \widetilde{g^1_m(h)}(t)|+ \\ \int_0^t L(s)|\widetilde{g^1_{m-1}(h)}(s) - \widetilde{g^1_m(h)}(s)|ds) \leq M(t)|I^pM|q^1_m(h) = M(t)(q^1_{m+1}(h) - s_m) < M(t)q^1_{m+1}(h), \end{gathered} \tag{6}$$

where $s_m := |I^pM|^m(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $h \in L^1(I_1,\mathbf{R})$, we put

$$\Psi^1_{m+1}(h) = \{f \in L^1(I_1,\mathbf{R});\ \ f(t) \in F(t,\widetilde{g^1_m(h)}(t),V(\widetilde{g^1_m(h)})(t)) \quad a.e.\ (I_1)\},$$

$$\Phi^1_{m+1}(h) = \begin{cases} \{h\} & \text{if } h \in \mathscr{S}_{I_1}(x_0,x_1), \\ \Psi^1_{m+1}(h) & \text{otherwise.} \end{cases}$$

Again, Lemma 3.2 (applied for $\phi(h) = g^1_m(h)$) allows to conclude that $\Phi^1_{m+1}(.)$ is is l.s.c. with nonempty closed decomposable values. At the same time, from (6), if $h \in L^1(I_1,\mathbf{R})$, the set

$$H^1_{m+1}(h) = cl\{f \in \Phi^1_{m+1}(h); \quad |f(t) - g^1_{m+1}(h)(t)| < M(t)q^1_{m+1}(h) \quad a.e.\ (I_1)\}$$

is nonempty. As above, via Lemma 2.2, it is obtained a selection g^1_{m+1} of H^1_{m+1} continuous with (a_1)-(d_1).

We conclude that

$$|g^1_{m+1}(h) - g^1_m(h)|_{1,1} \leq |I^pM|^m(\frac{1}{\Gamma(p)}|q^1_0(h)|_{1,1} + \varepsilon)$$

which means that the sequence $\{g^1_m(h)\}_{m\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $L^1(I_1,\mathbf{R})$. Take $g^1(h) \in L^1(I_1,\mathbf{R})$ its limit. Since the mapping $s \to |q^1_0(h)|_{1,1}$ is continuous, thus it is locally bounded and the Cauchy condition is satisfied by $\{g^1_m(h)\}_{m\in\mathbf{N}}$ locally uniformly with respect to h. Therefore, $g^1(.) : L^1(I_1,\mathbf{R}) \to L^1(I_1,\mathbf{R})$ is continuous.

Taking into account (a_1) we find that $g^1(h) = h, \quad \forall h \in \mathscr{S}_{I_1}(x_0,x_1)$ and from the hypotheses that the values of F are closed and (b_1) we find that

$$g^1(h)(t) \in F(t,\widetilde{g^1(h)}(t),V(\widetilde{g^1(h)})(t)), \quad a.e.\,(I_1) \quad \forall h \in L^1(I_1,\mathbf{R}).$$

At the final step of the induction procedure we assume that $g^i(.) : L^1(I_i, \mathbf{R}) \to L^1(I_i, \mathbf{R}),\ i = 2,...,k-1$ are constructed and satisfying (I)-(III) and we construct $g^k(.) : L^1(I_k, \mathbf{R}) \to L^1(I_k, \mathbf{R})$ continuous with (I)-(III).

We introduce the map $g_0^k : L^1(I_k, \mathbf{R}) \to L^1(I_k, \mathbf{R})$

$$g_0^k(h)(t) = g^{k-1}(h|_{I_{k-1}})(t)\chi_{I_{k-1}} + h(t)\chi_{I_k \setminus I_{k-1}}(t) \tag{7}$$

Since $g^{k-1}(.)$ is continuous and for $h_0, h \in L^1(I_k, \mathbf{R})$ we have

$$|g_0^k(h) - g_0^k(h_0)|_{1,k} \le |g^{k-1}(h|_{I_{k-1}}) - g^{k-1}(h_0|_{I_{k-1}})|_{1,k-1} + \int_{k-1}^{k} |h(t) - h_0(t)|dt,$$

and we deduce that $g_0^k(.)$ is continuous.

At the same time, the equality $g^{k-1}(h) = h,\ \forall h \in \mathscr{S}_{I_{k-1}}(x_0, x_1)$ and (7) allow to obtain

$$g_0^k(h) = h, \quad \forall h \in \mathscr{S}_{I_k}(x_0, x_1).$$

For $h \in L^1(I_k, \mathbf{R})$, we define

$$\Psi_1^k(h) = \{l \in L^1(I_k, \mathbf{R});\ l(t) = g^{k-1}(h|_{I_{k-1}})(t)\chi_{I_{k-1}}(t) + n(t)\chi_{I_k \setminus I_{k-1}}(t),\ n(t) \in F(t, \widetilde{g_0^k(h)}(t), V(\widetilde{g_0^k(h)})(t))\ a.e.\ ([k-1,k])\}$$

$$\Phi_1^k(h) = \begin{cases} \{h\} & \text{if } h \in \mathscr{S}_{I_k}(x_0, x_1), \\ \Psi_1^k(h) & \text{otherwise.} \end{cases}$$

Once again Lemma 3.2 (applied for $\phi(h) = g_0^k(h)$) implies that $\Phi_1^k(.) : L^1(I_k, \mathbf{R}) \to \mathscr{D}(I_k, \mathbf{R})$ is l.s.c.. In addition, if $h \in L^1(I_k, \mathbf{R})$ one may write

$$d(g_0^k(t), F(t, \widetilde{g_0^k(h)}(t), V(\widetilde{g_0^k(h)})(t)) = d(h(t), F(t, \widetilde{g_0^k(h)}(t), V(\widetilde{g_0^k(h)}(t))\chi_{I_k \setminus I_{k-1}} \le q_0^k(h)(t) \quad a.e.\ (I_k), \tag{8}$$

where

$$q_0^k(h)(t) = |h(t)| + q(t) + L(t)(|\widetilde{g_0^k(h)}(t)| + \int_0^t L(s)|\widetilde{g_0^k(h)}(s)|ds).$$

Obviously, $q_0^k : L^1(I_k, \mathbf{R}) \to L^1(I_k, \mathbf{R})$ is continuous. If $m \ge 0$ we define

$$q_{m+1}^k(h) = |I^p M|^m (\frac{k^{p-1}}{\Gamma(p)} |q_0^k(h)|_{1,k} + \varepsilon_{m+1}).$$

and from the continuity of $q_0^k(.)$ we deduce the continuity of $q_m^k : L^1(I_k, \mathbf{R}) \to L^1(I_k, \mathbf{R})$.

Finally, we provide the existence of $g_m^k : L^1(I_k, \mathbf{R}) \to L^1(I_k, \mathbf{R})$ continuous such that

$$g_m^k(h)(t) = g^{k-1}(h|_{I_{k-1}})(t) \quad \forall t \in I_{k-1}, \tag{a_k}$$

$$g_m^k(h) = h \quad \forall h \in \mathscr{S}_{I_k}(x_0, x_1), \tag{b_k}$$

$$g_m^k(h)(t) \in F(t, \widetilde{g_{m-1}^k(h)}(t), V(\widetilde{g_{m-1}^k(h)})(t)) \quad a.e.\ (I_k), \tag{c_k}$$

$$|g_1^k(h)(t) - g_0^k(h)(t)| \le q_0^k(h)(t) + \varepsilon_0 \quad a.e.\ (I_k), \tag{d_k}$$

$$|g_m^k(h)(t) - g_{m-1}^k(h)(t)| \le M(t) q_{m-1}^k(h) \quad a.e.\ (I_k), \quad m \ge 2. \tag{e_k}$$

Set

$$H_1^k(h) = cl\{f \in \Phi_1^k(h); \quad |f(t) - g_0^k(h)(t)| < q_0^k(h)(t) + \varepsilon_0 \quad a.e.\ (I_k)\}.$$

Using (8), $H_1^k(h) \ne \emptyset$ for any $h \in L^1(I_1, \mathbf{R})$. Taking into account Lemma 2.2 and the fact that the maps g_0^k, q_0^k are continuous we find a continuous selection g_1^k of H_1^k with (a_k)-(d_k).

If $g_i^k(.),\ i = 1, \dots m$ with (a_k)-$(e_k)$are already constructed, from (e_k) one may write

$$\begin{gathered} d(\underbrace{g_m^k(h)(t), F}(t, \widetilde{g_m^k(h)}(t), V(\widetilde{g_m^k(h)})(t)) \le L(t)(|\widetilde{g_{m-1}^k}(h)(t) - \widetilde{g_m^k}(h)(t)| + \\ \int_0^t L(s)|\widetilde{g_{m-1}^k}(h)(s) - \widetilde{g_m^k}(h)(s)|ds) \le M(t)(q_{m+1}^k(h) - s_m) < M(t) q_{m+1}^k(h), \end{gathered} \tag{9}$$

where $s_m := |I^p M|^m(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $h \in L^1(I_k, \mathbf{R})$, we define

$$\Psi^k_{m+1}(h) = \{l \in L^1(I_k,\mathbf{R});\ l(t) = g^{k-1}(\widetilde{h|_{I_{k-1}}})(t)\chi_{I_{k-1}}(t) + n(t)\chi_{I_k \setminus I_{k-1}}(t),\quad n(t) \in F(t, \widetilde{g^k_m(h)}(t), V(\widetilde{g^k_m(h)})(t))\quad a.e.\ ([k-1,k])\},$$

$$\Phi^k_{m+1}(h) = \begin{cases} \{h\} & \text{if } h \in \mathscr{S}_{I_k}(x_0,x_1), \\ \Psi^k_{m+1}(h) & \text{otherwise.} \end{cases}$$

Applying Lemma 3.2 it is obtained that $\Phi^k_{m+1}(.) : L^1(I_k,\mathbf{R}) \to \mathscr{P}(L^1(I_k,\mathbf{R}))$ has nonempty closed decomposable values and is l.s.c.. As above, the set

$$H^k_{m+1}(h) = cl\{f \in \Phi^k_{m+1}(h);\ |f(t) - g^k_{m+1}(h)(t)| < M(t)q^k_{m+1}(h)\quad a.e.\ (I_k)\}\quad h \in L^1(I_k,\mathbf{R})$$

is nonempty. Again, Lemma 2.2 allows to obtain a continuous selection g^k_{m+1} of H^k_{m+1}, verifying (a_k)-(e_k).

By (e_k) one has

$$|g^k_{m+1}(h) - g^k_m(h)|_{1,k} \le |I^p M|^m [\frac{k^{p-1}}{\Gamma(p)} |q^k_0(h)|_{1,1} + \varepsilon].$$

Repeating the proof done in the first case we get the convergence of $\{g^k_m(h)\}_{m\in\mathbf{N}}$ to some $g^k(h) \in L^1(I_k,\mathbf{R})$. Moreover, $g^k(.) : L^1(I_k,\mathbf{R}) \to L^1(I_k,\mathbf{R})$ is continuous.

By (a_k) we have that

$$g^k(h)(t) = g^{k-1}(h|_{I_{k-1}})(t)\quad \forall t \in I_{k-1},$$

by (b_k) $g^k(h) = h$ $\forall h \in \mathscr{S}_{I_k}(x_0,x_1)$ and, finally, since the values of F are closed, from (c_k) we deduce that

$$g^k(h)(t) \in F(t, \widetilde{g^k(h)}(t), V(\widetilde{g^k(h)})(t)),\quad a.e.\ (I_k)\quad \forall h \in L^1(I_k,\mathbf{R}),$$

and the proof is complete.

Remark 3.4. By definition, a subspace X of a Hausdorff topological space Y is said to be a retract of Y if there exists a mapping $h : Y \to X$ continuous with $h(x) = x$, $\forall x \in X$.

So, Theorem 3.3 states that for each $x_0, x_1 \in \mathbf{R}$, the set $\mathscr{S}(x_0,x_1)$ is a retract of the Banach space $L^1_{loc}([0,\infty),\mathbf{R})$.

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